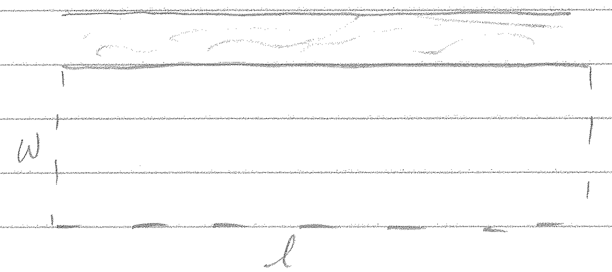
14 Word Problems, Algebraic and Trigonometric

Mumble some words about optimization.

Eg: ① A rectangular field is to be fenced off next to a straight river, with fencing on three sides with the river marking off the fourth side. Exactly 100 feet of fencing is to be used. Express the area of the field as a function of the width.

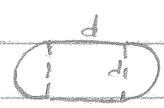


$A(w,l) = 2w + l = 100 \Rightarrow l = 100 - 2w$

$P(w,l) = wl$

$P(w, 100-2w) = w(100-2w) = 100w - 2w^2.$

② A swimming pool has the shape of a square with a semicircle at each end. Express the perimeter and area of the pool as a function of the diameter of the semicircles.



Perimeter of a circle: $2\pi r$

Perimeter of the square: $2d$

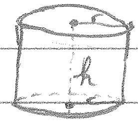
Area of a circle: πr^2

Area of a square: d^2

$$A = \pi r^2 + d^2 = \pi\left(\tfrac{d}{2}\right)^2 + d^2 = \pi \, d^2/4 + d^2$$

$$P = 2\pi r + 2d = 2\pi\left(\tfrac{d}{2}\right) + 2d = \pi d + 2d$$

③ A cylindrical tin can has height h cm and radius r cm. Its volume is 32 cm³. Express h as a function of r, and vise-versa

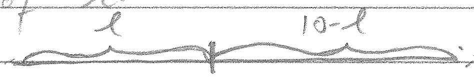


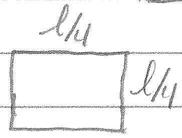
$V = h\pi r^2$

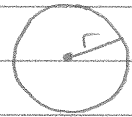
$32 = h\pi r^2 \Rightarrow h = 32/\pi r^2$ cm

$r = \sqrt{32/h\pi}$ cm

④ A 10-inch wire is cut into two pieces. One piece has length l, is bent into a square. The other piece is bent into a circle. What is the total area of the two shapes as a function of l.







$P = 2\pi r$

$10 - l = 2\pi r$

$\Rightarrow r = (10-l)/2\pi$

$A_s = l^2/16$

$A_c = \pi r^2 = \pi(10-l)^2/(2\pi)^2$

$= \dfrac{(10-l)^2}{4\pi}$

$A = A_s + A_c$

$= l^2/16 + (10-l)^2/4\pi.$

Def^n ① y is proportional to x if $y = kx$, k the proportionality constant.

② y is inversely proportional to x if $y = k/x$.

③ y is said to follow the inverse square law with respect to x if $y = k/x^2$.

Eg ⑤ The cost C of building a highway is proportional to its length. A 7.5-mile section costs \$4 million. Express the cost as a function of length and compute the cost of 13.2 miles

$C(\ell) = k\ell$, given $C(2.5) = 1 = k2.5 = k(5/2)$
$\Rightarrow k = 2/5.$
$C(13.2) = C(\frac{66}{5}) = 2/5(66/5) = 132/25 = \$5.28$ million

⑥ The gravitational force between two point masses satisfies the inverse square law with respect to the distance between them. Suppose the gravitational force acting on you is 150 lbs. and that you and the earth can be considered point masses with mass concentrated at the center. If the radius of the earth is approximately 4000 miles and Everest is 29,028 feet high, estimate the gravitational force acting on you at the top of Everest.

$F = k/r^2$, r the radius of the earth.

$150 = k/(4000)^2$ $\qquad 29{,}028' \approx 5.5$ miles

$\Rightarrow k = 2.4 \times 10^9.$

$$\Rightarrow F \approx \frac{2.4 \times 10^9}{(4000+5.5)^2} = \frac{2.4 \times 10^9}{(4005.5)^2} \approx 149.6$$

⑦ Bob drives 60 mph for 100 miles. Alice waits W minutes after Bob leaves to make the same trip.

a) Express the speed S Alice must drive to arrive at the same time as Bob as a function of W.

$S = D/T \Rightarrow T = D/S$

$T_B = 100/60, \quad T_A = 100/S + W$

$100/60 = 100/S + W$

$\Rightarrow 100/S = 100/60 - W \Rightarrow 100 = S(100/60 - W) = S\left(\frac{100-60W}{60}\right) = S\left(\frac{5-3W}{3}\right)$

$\Rightarrow S = 100\left(\frac{3}{5-3W}\right) = 300/(5-3W)$

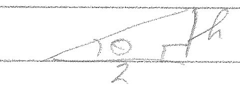
b) If Alice waits half an hour, how fast must she drive?

$$S = \frac{300}{5-3/2} = \frac{300}{\frac{7}{2}} = \frac{600}{7} \approx 86 \text{ mph}.$$

$$\begin{array}{r} 85.71 \\ 7\overline{)600} \\ 56 \\ \hline 40 \\ 35 \\ \hline 50 \\ 49 \\ \hline 10 \end{array}$$

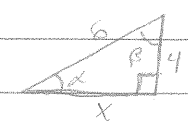
14.2 Solving Right Triangles

Eg: ① A rocket blasts off vertically, its path being followed by a camera on the ground 2 miles away. Find a relation between the height of the rocket and the angle of the camera.



$\tan(\theta) = h/2, \quad h = 2\tan(\theta).$

② Given the triangle

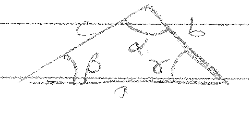


Find x, α, β.

$$x^2 = 6^2 - 4^2 = 36 - 16 = 20 \Rightarrow x = \sqrt{20} = 2\sqrt{5}$$
$$\alpha = \arcsin(2/3)$$
$$\beta = 90 - \alpha = 90 - \arcsin(2/3)$$

14.3 The Law of Sines and the Law of Cosines

Consider the triangle



Law of Sines: $\dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c}$.

Law of Cosines: $c^2 = a^2 + b^2 - 2ab\cos(\gamma)$

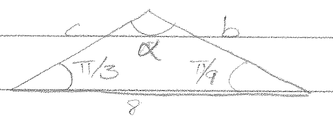
$a^2 = b^2 + c^2 - 2bc\cos(\alpha)$

$b^2 = a^2 + c^2 - 2ac\cos(\gamma)$.

If, say, $\gamma = \pi/2$, then $c^2 = a^2 + b^2 - 2ab\cos(\gamma)$

$= a^2 + b^2$.

Eg.: ① Given



Find α, c, b.

$$\pi = \alpha + \pi/3 + \pi/9$$

$$\Rightarrow \alpha = 9\pi/9 - 3\pi/9 - \pi/9$$

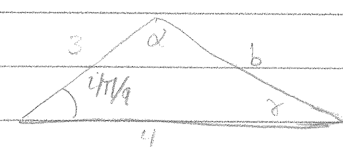
$$= 5\pi/9$$

$$\frac{\sin(5\pi/9)}{8} = \frac{\sin(\pi/3)}{b} = \frac{\sin(\pi/9)}{c}$$

$$\Rightarrow b = \frac{8\sin(\pi/3)}{\sin(5\pi/9)}$$

$$\Rightarrow c = \frac{8\sin(\pi/9)}{\sin(5\pi/9)}.$$

②



$$b^2 = 4^2 + 3^2 - 2(4)(3)\cos(4\pi/9)$$

$$\Rightarrow b = \sqrt{25 - 24\cos(4\pi/9)}$$

$$\frac{\sin(\alpha)}{4} = \frac{\sin(4\pi/9)}{\sqrt{25 - 24\cos(4\pi/9)}}$$

$$\Rightarrow \sin(\alpha) = \frac{4\sin(4\pi/9)}{\sqrt{25 - 24\cos(4\pi/9)}}$$

$$\Rightarrow \alpha = \arcsin\left(\frac{4\sin(4\pi/9)}{\sqrt{25 - 24\cos(4\pi/9)}}\right) \approx \pi/3$$

$$\gamma = \frac{9\pi}{9} - 4\pi/9 - \alpha$$

$$= 5\pi/9 - \alpha$$

$$\approx 2\pi/9$$