10 Changing the Form of a Function

10.1 Factor common terms

Eg. ① $3x^2y^3 + 15xy^4 - 21x^3y^2 = 3xy^2(xy + 5y^2 - 7x^2)$

10.2 Special Formulas

1) $x^2 - y^2 = (x+y)(x-y)$ (difference of two squares)
2) $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$
3) $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$
4) $x^2 + (a+b)x + ab = (x+a)(x+b)$
5) $acx^2 + (bc+ad)x + bd = (ax+b)(cx+d)$
6) $x^2 + 2xy + y^2 = (x+y)^2$ and $x^2 - 2xy + y^2 = (x-y)^2$

Eg. ① Difference of two squares

a) $z^2 - 9 = (z+3)(z-3)$,
b) $x^4 - y^2 = (x^2)^2 - y^2 = (x^2 - y)(x^2 + y)$,
c) $(x-y)^2 - 4y^2 = (x-y)^2 - (2y)^2$
$= ((x-y) - 2y)((x-y) + 2y)$
$= (x - 3y)(x + y)$.

② Sum of two cubes

a) $a^3 + 8b^3 = a^3 + (2b)^3 = (a+2b)(a^2 - (a)(2b) + (2b)^2)$
$= (a+2b)(a^2 - 2ab + 4b^2)$

b) $27x^3 + 64y^3z^6 = (3x)^3 + (4yz^2)^3$
$= (3x + 4yz^2)(9x^2 - 12xyz^2 + 16y^2z^4)$

c) $a^3 + 2b^3 = a^3 + (\sqrt[3]{2}\,b)^3 = (a + \sqrt[3]{2}\,b)(a^2 - \sqrt[3]{2}ab + \sqrt[3]{4}\,b^2)$.

③ Difference of two cubes

$x^3 - 64y^6 = x^3 - (4y^2)^3 = (x - 4y^2)(x^2 + 4xy^2 + 16y^4)$

(4) $x^2+5x+6 = x^2+(2+3)x+(2)(3)$
$= (x+2)(x+3)$

(5) $y^2-10y+21 = y^2-(7+3)y+(7)(3)$
$= (y-7)(y-3)$

(6) $s^2+2s-8 = s^2+(4+(-2))x+(4)(-2)$
$= (s+4)(s-2)$

(7) $a^4-a^2-6 = (a^2)^2-(a^2)-6$
$= (a^2)^2+(2+(-3))(a^2)+(2)(-3)$
$= (a^2+2)(a^2-3)$
$= (a+i\sqrt{2})(a-i\sqrt{2})(a+\sqrt{3})(a-\sqrt{3})$

(8) $y^2+10y-24 = y^2+(12+(-2))y+(12)(-2)$
$= (y+12)(y-2)$

16.3 Grouping

$5y(2x+3)+2(x+3)$

Eg. (1) factor $10xy+15y+4x+6$

$10xy+15y+4x+6 = 5y(2x+3)+2(2x+3)$
$= (2x+3)(5y+2)$.

(2) $6ax+3ay-4bx-2by+10x+5y$
$= 3a(2x+y)-2b(2x+y)+5(2x+y)$
$= (2x+y)(3a-2b+5)$.

Rmk: If an expression has a prime number of terms, then factoring by grouping won't work!

Why? Say we have $t_1+\dots+t_p$ and under some grouping we have
$t_1+\dots+t_p = g t_1' + g t_2' + \dots + g t_s'$, $1 < s < p$
$= g(t_1'+t_2'+\dots+t_s')$.

Then q has some number of terms, say r, and the number of terms in the product is $rs = p$ — but the only divisors of p are 1 or p, a contradiction since $r, s < p$.

10.4 Factor Theorem

Let $P(x)$ be a polynomial. Let a be a real number. Then $x-a$ is a factor of $P(x)$ if and only if $P(a)=0$.

Eg.: ① Factor $P(x) = x^3 - 2x^2 - 5x + 6$.

$P(1) = 1 - 2 - 5 + 6 = 7 - 7 = 0$.
So $(x-1)$ is a factor of $P(x)$.

Polynomial Division

$$
\begin{array}{r|l}
 & \;\; x^2 - x - 6 \\
\hline
x-1 & x^3 - 2x^2 - 5x + 6 \\
 & \underline{-x^3 + x^2} \\
 & \quad -x^2 - 5x \\
 & \quad \underline{+x^2 - x} \\
 & \qquad -6x + 6 \\
 & \qquad \underline{+6x - 6} \\
 & \qquad\quad 0
\end{array}
$$

$$\Rightarrow x^3 - 2x^2 - 5x + 6 = (x-1)(x^2 - x - 6).$$
$$= (x-1)(x+3)(x-2).$$

② $2x^2 + 3x - 2$

Use the factor theorem and the quadratic equation.

$$x = \frac{-3 \pm \sqrt{9+16}}{4} = \frac{-3 \pm \sqrt{25}}{4} = \frac{-3 \pm 5}{4}$$

$x = \frac{1}{2}$ or $x = -2$

so we have

$$2x^2+3x-2=(x-\tfrac{1}{2})(x+2)(\_\_)$$

but the only thing missing is a 2, so that must be the missing piece.

$$2(x-\tfrac{1}{2})(x+2)=(2x-1)(x+2)$$
$$=2x^2+4x-x-2$$
$$=2x^2+3x-2. \blacksquare$$

③ Can x^2+x+1 be factored?
Not over the reals, no. $\mathrm{Disc}(x^2+x+1)=1-4=-3$.

$$x=\frac{-1\pm\sqrt{-3}}{2}=\frac{-1\pm i\sqrt{3}}{2}$$

$$\rightsquigarrow \left(-\tfrac{1}{2},\tfrac{\sqrt{3}}{2}\right)=\left(\cos(2\pi/3),\sin(2\pi/3)\right) \text{ or}$$
$$\left(-\tfrac{1}{2},-\tfrac{\sqrt{3}}{2}\right)=\left(\cos(4\pi/3),\sin(4\pi/3)\right).$$

$x^3-1=(x-1)(x^2+x+1)$ "Third roots of unity."

10.5 Rationalizing Numerators or Denominators Using Conjugates.

Consider $x-\sqrt{2}$. Apply the difference of two squares in reverse:

$$(x-\sqrt{2})(x+\sqrt{2})=x^2-(\sqrt{2})^2=x^2-2.$$

The expression $x+\sqrt{2}$ is the conjugate of $x-\sqrt{2}$ (and vise-versa).

Eg. ①: $\frac{(x^2-3)(x-\sqrt{3})}{(x+\sqrt{3})(x-\sqrt{3})} = \frac{(x^2-3)(x-\sqrt{3})}{x^2-3} = x-\sqrt{3}$.

② $\frac{x^4-25}{x-\sqrt{5}} = \frac{(x^2-5)(x^2+5)}{x-\sqrt{5}} \frac{(x+\sqrt{5})}{(x+\sqrt{5})}$

$= \frac{\cancel{(x^2-5)}(x^2+5)(x+\sqrt{5})}{\cancel{x^2-5}}$

$= (x^2+5)(x+\sqrt{5})$

③ $\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)\frac{(\sqrt{x+h}+\sqrt{x})}{(\sqrt{x+h}+\sqrt{x})} = \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} = \frac{1}{\sqrt{x+h}+\sqrt{x}}$.

Rmk: This is the major portion of the calculation of the derivative of $f(x)=\sqrt{x}$.

① Write $1/\sqrt{x+h} - 1/\sqrt{x}$ as a fraction and rationalize the numerator:

$\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}} = \frac{\sqrt{x}-\sqrt{x+h}}{(\sqrt{x})(\sqrt{x+h})} = \frac{x-(x+h)}{(\sqrt{x})(\sqrt{x+h})(\sqrt{x}+\sqrt{x+h})}$

$= \frac{-h}{(\sqrt{x})(\sqrt{x+h})(\sqrt{x}+\sqrt{x+h})}$

10.6 Extracting Factors from Radicals

Eg: ① Simplify $\sqrt[3]{250x^4y^3}$

$\sqrt[3]{250x^4y^3} = \sqrt[3]{2\cdot 5^3\cdot x^3\cdot y^3\cdot x}$

$= 5xy\sqrt[3]{2x}$.

② $\sqrt{25x^8y} = 5x^4\sqrt{y}$.

③ $\sqrt{36xy^2z^3} = 6|y||z|\sqrt{xz}$

Rmk: $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

If $x<0$, then $-x>0$, so $\sqrt{x^2} = \sqrt{(-x)^2} = -x$
$-x$ is the positive number that squares to x^2. If $x>0$, then

$$\sqrt{x^2} = x$$

since x is the positive number that squares to x^2. Therefore

$$\sqrt{x^2} = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0 \end{cases}$$

implies that $\sqrt{x^2} = |x|$. ∎

④ Simplify

$$\sqrt[4]{16x^8y^2} = 2x^2\sqrt[4]{y^3}.$$

⑤ $\sqrt[5]{32x^{10}y^2} = 2x^2\sqrt[5]{y^2}.$

⑥ $\sqrt{x^2y^6+3x^5y^4} = \sqrt{x^2y^4(y^2+3x^3)} = |x|y^2\sqrt{y^2+3x^3}.$