8 Logarithmic Functions

Def^{n}: Let $0<a\neq 1$ be a real number. Then
$y=\log_a(x) \iff a^y=x$. (inverse of a^x).

E.g.: ① $\log_2(8)=3$

$8=2^3$, so $\log_2(8)=\log_2(2^3)=3$.

② $\log_{10}(1000\,000)=\log_{10}(10^6)=6$.

③ $\log_{10}(0.01)=\log_{10}(1/100)=\log_{10}(1/10^2)=\log_{10}(10^{-2})=-2$.

④ $\log_2(32)=\log_2(2^5)=5$.

⑤ $\log_3(81)=\log_3(3^4)=4$.

⑥ $\log_7(7^{15})=15$.

⑦ $\log_a(a^3)=3$.

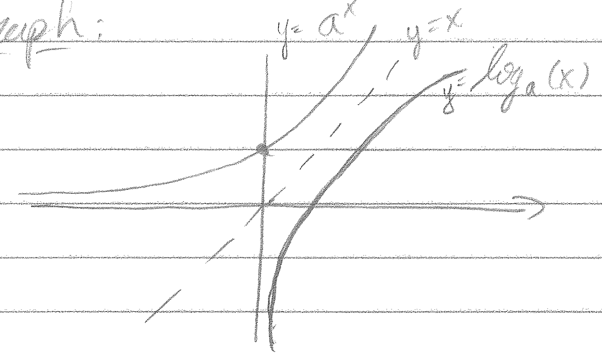
8.1 Logs as Inverses of Exponential Functions

Let $0<a\neq 1$ be given. Then $\log_a(x)$ and a^x are inverses of one another.

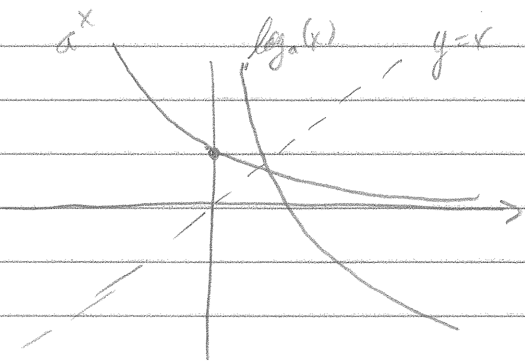
Let $f(x)=a^x$ and $g(x)=\log_a(x)$

$$(f\circ g)(x)=a^{\log_a(x)}=x,$$
$$(g\circ f)(x)=\log_a(a^x)=x.$$

Graph:



$a > 1$



$0 < a < 1$

Eg: ① Solve $\log_2(x) = 4$

$\Rightarrow 2^{\log_2(x)} = 2^4$

$\Rightarrow x = 2^4$

$\Rightarrow x = 16.$

② Solve $\log_{10}(x) = 3$

$\Rightarrow 10^{\log_{10}(x)} = 10^3$

$\Rightarrow x = 1000$

③ Solve $\log_{10}(x^2 - 4x + 14) = 1$

$\Rightarrow x^2 - 4x + 14 = 10$

$\Rightarrow x^2 - 4x + 4 = 0$

$\Rightarrow (x-2)^2 = 0$

$\Rightarrow x = 2.$

④ Solve $\log_3(x^2-3x-7)=1$

$x^2-3x-7=3$

$\Rightarrow x^2-3x-10=0$

$\Rightarrow (x-5)(x+2)=0$

$\Rightarrow x=5$ or $x=-2$

⑤ $2^{x^2+1}=8$

$\log_2(2^{x^2+1}) = \log_2(8)=3$

$\Rightarrow x^2+1=3$

$\Rightarrow x^2-2=0$

$\Rightarrow x=\pm\sqrt{2}$

⑥ $100^{\sin(x)}=10, \quad x\in[0, \pi/2]$.

$\Rightarrow \sin(x) = \log_{100}(10) = \log_{10^2}(10) = 1/2$

$\Rightarrow x=\pi/6$.

8.3 Laws of Logarithms

Let $0<a\neq 1$ be a real number and $x,y>0$ be given

1) $\log_a(xy) = \log_a(x) + \log_a(y)$,
2) $\log_a(x/y) = \log_a(x) - \log_a(y)$,
3) $\log_a(x^r) = r\log_a(x)$,
4) $\log_a(1)=0$,
5) $\log_a(x) = \log_b(x)/\log_b(a)$

Eg. ① Solve $\log_2(x^2) + \log_2(2x) = 4$

$4 = \log_2(x^2 \cdot 2x) = \log_2(2x^3)$
$\Rightarrow 16 = 2x^3$
$\Rightarrow x^3 = 8$
$\Rightarrow x = 2.$

② $\log_{10}((x^2-3x)^3) = 3$

$3 = \log_{10}((x^2-3x)^3) = 3\log_{10}(x^2-3x)$
$\Rightarrow \log_{10}(x^2-3x) = 1$
$\Rightarrow x^2-3x = 10$
$\Rightarrow x^2-3x-10 = 0$
$\Rightarrow (x-5)(x+2) = 0$
$\Rightarrow x = 2, \ x = 5.$

③ $\log_2(x) + \log_2(x-1) = 1$
$\Rightarrow \log_2(x^2-x) = 1$
$\Rightarrow x^2-x = 2$
$\Rightarrow x^2-x-2 = 0$
$\Rightarrow (x-2)(x+1) = 0$
$\Rightarrow x = 2, -1$
but $\log_2(-1)$ is not defined, so $x=2$ is the only solution

④ $\log_8(2) = \dfrac{\log_2(2)}{\log_2(8)} = \dfrac{1}{3}$