7 Composition and Inverse Functions

7.1 Composition

Def^{n}: Let $f(x)$ and $g(x)$ be functions. The composition of f with g is $(f \circ g)(x) = f(g(x))$.

Eg. ① $f(x) = x^2 + 2x + \pi$, $g(x) = x^3$.

$$(f \circ g)(x) = f(g(x)) = f(x^3) = (x^3)^2 + 2(x^3) + \pi$$
$$= x^6 + 2x^3 + \pi.$$

② $f(x) = \tan(x)$, $g(x) = \sqrt{x+1}$

$$\left.\begin{array}{l}(f \circ g)(x) = \tan(\sqrt{x+1}) \\ (g \circ f)(x) = \sqrt{\tan(x)+1}\end{array}\right\}$$ not abelian!

③ $f(x) = 1/x$, $g(x) = \sin(x)$
$(f \circ g)(x) = 1/\sin(x) = \csc(x)$.

④ $f(x) = 1/x+2$, $g(x) = x^2 - 1$

$(f \circ g)(x) = \frac{1}{x^2+1}$
$(g \circ f)(x) = \left(\frac{1}{x+2}\right)^2 - 1 = \frac{1}{(x+2)^2} - 1$

⑤ $f(x) = x^2$, $g(x) = \sin(x)$, $h(x) = 2x+1$

$(f \circ g \circ h)(x) = \sin^2(2x+1)$.

⑥ $f(x) = x^3$, $g(x) = \cos(x)$, $h(x) = \sqrt{x}$, $k(x) = x+2$.
$f \circ g \circ h \circ k = \cos^3(\sqrt{x+2})$.

⑦ $f(x) = \cos(x)$, $g(t) = t^2$
$(f \circ g)(t) = \cos(t^2)$ (change of variables)

7.2 The Idea of Inverses

Def^n: Let $f(x)$ and $g(x)$ be functions. We say f and g are inverses of one another if

$$f \circ g = 1 \quad \text{and} \quad g \circ f = 1 \qquad (f(g(x)) = x, g(f(x)) = x).$$

We denote this relationship by $g = f^{-1}$ or $f = g^{-1}$.

Rmk: ① this is not a power. $f^{-1}(x) \neq \frac{1}{f(x)}$.

② The domains here are important.

$f(x) = x^2$ is invertible over $[0, \infty)$.

$f^{-1}(x) = \sqrt{x}$ (There is exactly one positive square root of any real number). However, this is not the case over $(-\infty, \infty)$; f does not have an inverse here!

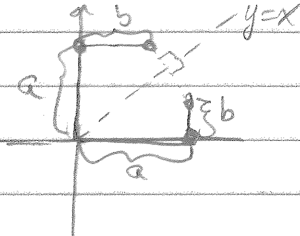
$$\sqrt{(-2)^2} = \sqrt{4} = 2 \neq -2$$

7.3 Finding the Inverse of f Given by a Graph

Let f be a function. Its graph is the set of points $(x, f(x))$. The graph of f^{-1} are the set of points

$$(f(x), f^{-1} \circ f(x)) = (f(x), x).$$

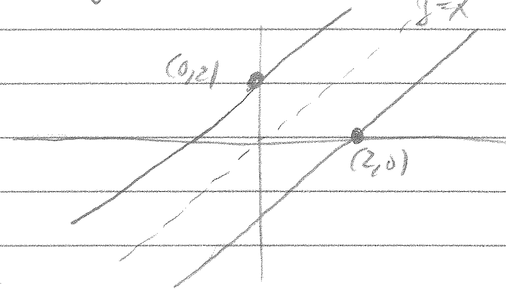
So the graph of f^{-1} is the graph of f with the co-ordinates reversed.



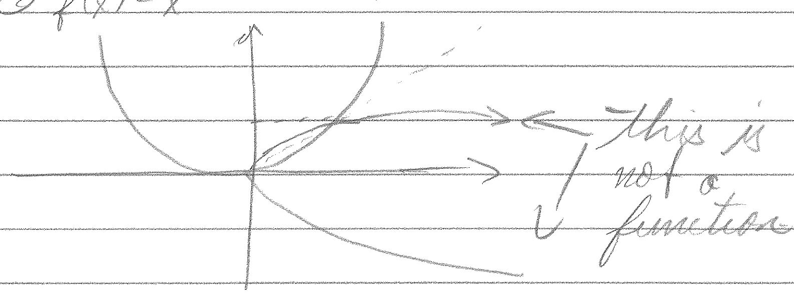
This is a reflection across the line $y = x$.

E.g. ① $f(x) = x+2, \; g(x) = x-2.$

$$f(g(x)) = f(x-2) = (x-2)+2 = x$$
$$g(f(x)) = g(x+2) = (x+2)-2 = x.$$



② $f(x) = x^2$



Mention the Horizontal Line Test.

7.4 Finding the Inverse of f Given by an Expression

1. Write down $y = f(x)$,
2. Solve for x, this is $f^{-1}(y) = x$.
3. Interchange x any y

E.g. ① $f(x) = \frac{2}{x+3}$.

$$y = \frac{2}{x+3} \Rightarrow x+3 = 2/y \Rightarrow x = \frac{2}{y} - 3$$
$$\Rightarrow x = \frac{2-3y}{y}$$

$$\Rightarrow f^{-1}(x) = \frac{2-3x}{x}$$

* Check $f \circ f^{-1} = f^{-1} \circ f = 1$.

② Invert $f(x) = \sqrt[3]{2x+1}$

$y = \sqrt[3]{2x+1}$

$\Rightarrow y^3 = 2x+1$

$\Rightarrow \frac{y^3-1}{2} = x$

$= f^{-1}(x) = \frac{x^3-1}{2}$

③ Things go poorly....

$f(x) = x^4 - 3$

$y = x^4 - 3$

$\Rightarrow y+3 = x^4$

$\Rightarrow \pm\sqrt[4]{y+3} = x$

$f^{-1}(x) = \pm\sqrt[4]{x+3}$ is not a function!