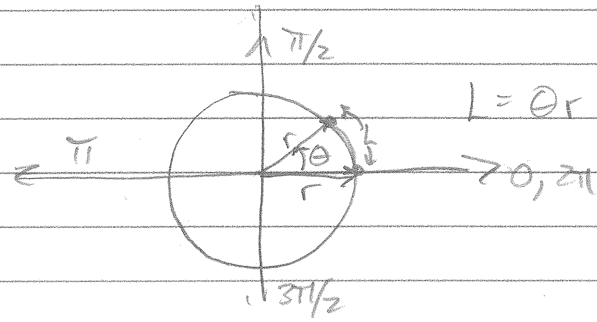
5 Cyclic Phenomena: The Six Basic Trigonometric Functions

5.1 Angles

The unit circle is a circle of radius 1 about the origin, (0,0). It is given by the equation

$$x^2+y^2=1$$



Angles (positive) are measured counter clockwise, negative angles are measured clockwise

Circumference of a circle: $2\pi r$

$360° = 2\pi$

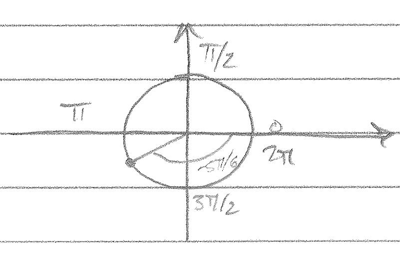
$\Rightarrow 1° = 2\pi/360$

Eg: ① Convert 27° to radians:

$$27 \cdot \frac{2\pi}{360} = \frac{3^3 \cdot 2\pi}{180 \cdot 2} = \frac{3^3 \cdot \pi}{9 \cdot 20} = \frac{3\pi}{20}$$

② Convert $-5\pi/6$ to degrees and draw it

$$-5\pi/6 = \frac{-5\pi}{6} \cdot \frac{360}{2\pi} = \frac{-5 \cdot 60}{2} = -300/2 = -150°$$



5.2 Definition of sin and cos

The points on the unit circle have x-coordinate $\cos(\theta)$ and y-coordinate $\sin(\theta)$.

Rmk: ① Since these points lie on $x^2+y^2=1$, we obtain the familiar identity

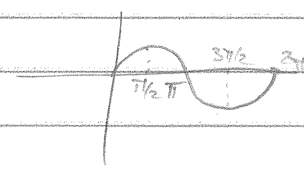
$$\sin^2(\theta)+\cos^2(\theta)=1.$$

② $\left.\begin{array}{l}\cos(\theta)^n=\cos^n(\theta)\\ \sin(\theta)^n=\sin^n(\theta)\end{array}\right\}$ notation

③ As θ varies through $[0,\pi/2]$

$\sin(0)=0 \nearrow \sin(\pi/2)=1$

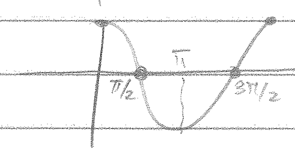
$\cos(0)=1 \searrow \cos(\pi/2)=0$



through $[\pi/2,\pi]$

$\sin(\pi/2)=1 \searrow \sin(\pi)=0$

$\cos(\pi/2)=0 \searrow \cos(\pi)=1$



through $[\pi,3\pi/2]$

$\cos(\pi)=1 \nearrow \cos(3\pi/2)=0$

$\sin(\pi)=0 \searrow \sin(3\pi/2)=-1$

through $[3\pi/2,2\pi]$

$\sin(3\pi/2)=-1 \nearrow \sin(2\pi)=0$

$\cos(3\pi/2)=0 \nearrow \cos(2\pi)=1.$

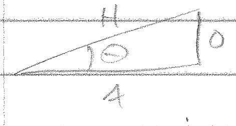
④ sin and cos are 2π-periodic (trace the circle).

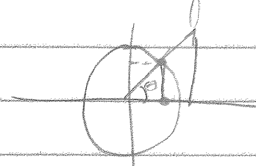
⑤ These are sinusoidal.

Eg: ① Evaluate $\cos(3\pi/2)$ and $\sin(3\pi/2)$

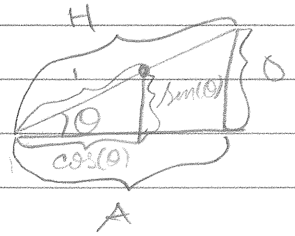
② Evaluate $\cos(9\pi/2)$.

$S=\frac{O}{H}$ $C=\frac{A}{H}$ $T=\frac{O}{A}$ (Alternative definition)





Similar triangles

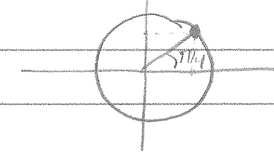


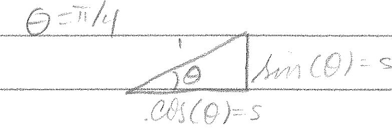
$$\frac{\sin(\theta)}{1} = \frac{O}{H} \Rightarrow \sin(\theta) = O/H$$

$$\frac{\cos(\theta)}{1} = \cos(\theta) = \frac{A}{H}$$

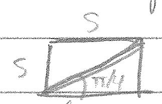
5.3 Special Angles ($\pi/4$, $\pi/6$, and $\pi/3$)

Eg: ① find $\sin(\pi/4)$.





The line of the triangle bisects the square



so we know $\sin(\pi/4) = \cos(\pi/4) = s$ and

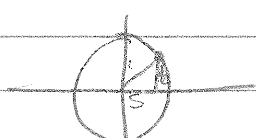
$1 = \cos^2(\frac{\pi}{4}) + \sin^2(\frac{\pi}{4}) = s^2 + s^2 = 2s^2$

$\Rightarrow s^2 = 1/2$

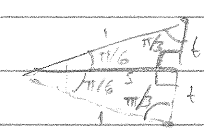
$\Rightarrow s = \frac{1}{\sqrt{2}}$.

$\Rightarrow \sin(\pi/4) = 1/\sqrt{2}$. ■

② find $\sin(\pi/6)$ and $\cos(\pi/6)$.



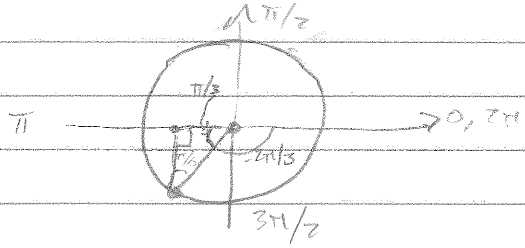
$t = \sin(\pi/6)$
$s = \cos(\pi/6)$



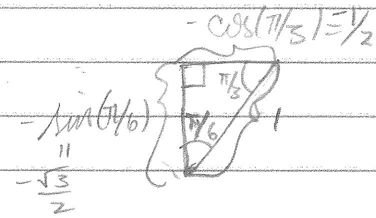
is equilateral, so $2t = 1 \Rightarrow t = 1/2$
$\Rightarrow \cos(\pi/6) = 1/2$

$\Rightarrow 1 = s^2 + t^2 = s^2 + 1/4$
$\Rightarrow s^2 = 1 - 1/4 = 3/4$
$\Rightarrow s = \sin(\pi/6) = \sqrt{3}/2$. ■

③ Find $\sin(-2\pi/3)$

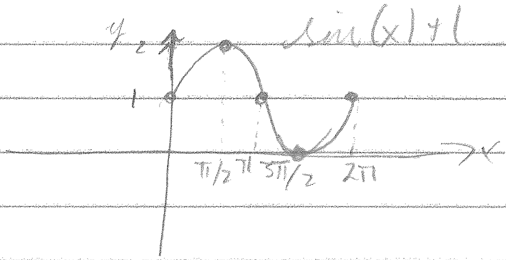


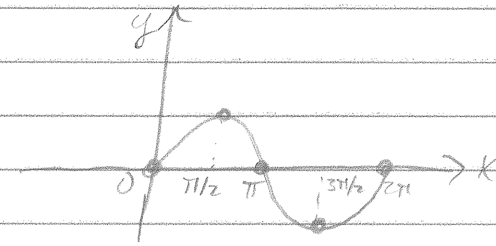
$$\pi - 2\pi/3 = \frac{3\pi}{3} - \frac{2\pi}{3} = \pi/3$$



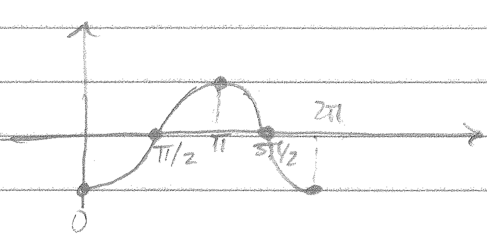
5.4 Graphs involving sin and cos

E.g. ① On $[0, 2\pi]$, graph $\sin(x)$, $1 + \sin(x)$, and $\sin(x - \pi/2)$.





$\sin(x - \pi/2)$



Rmk: This is called a phase shift.

Stretching/Compressing

Vertical: Let $f(x)$ be a function and $a>1$ a real number. Then the graph of $af(x)$ is a vertical stretching of $f(x)$.

Eg.: $f(x) = \sin(x)$, $a=2$
$2f(x) = 2\sin(x)$.

If $0<a<1$, then the graph of $af(x)$ is a compression of $f(x)$ by a.

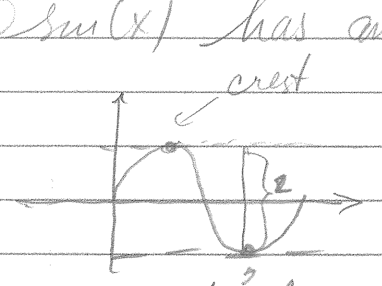
Eg.: $f(x) = \sin(x)$, $a = 1/2$.

Reflection: The graph of $-f(x)$ is the reflection of $f(x)$ across the x-axis. For $a<0$, the graph of $af(x)$ is the combination of a reflection and a stretching/compression.

Eg: $-\sin(x)$, $-\frac{1}{2}\sin(x)$, $-2\sin(x)$.

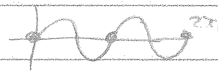
Def^n: The amplitude of a sinusoidal graph is half the distance between the crest and trough.

E.g. ① $\sin(x)$ has amplitude 1



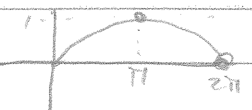
② $f(x) = 1-3\sin(x)$ has amplitude 3: it has the same amplitude as $3\sin(x)$ as it is just a vertical shift of $3\sin(x)$.

Horizontal: Let $f(x)$ be a function and $a>1$ a real number. Then $f(ax)$ is a horizontal compression of $f(x)$.

E.g.: $\sin(2x)$: $[0,\pi] \xrightarrow{2} [0,4\pi] \xrightarrow{\sin}$ 

Say sin has a oscillations if $a \in \mathbb{Z}$.

If $0<a<1$, then $f(ax)$ is a horizontal stretching of $f(x)$.

E.g. $\sin(x/2)$ 

This is one half a complete oscillation.

If f is an odd function ($f(-x) = -f(x)$), then for $a<0$, $f(-x)$ is a reflection about the y-axis. Here $f(-ax) = -f(ax)$ is a reflection of the horizontal stretching/compression.

Def^n: The length of a cycle is the period, T. The frequency is $1/T$.

Eg: $\sin(3x)$ has period $2\pi/3$ and frequency $3/2\pi$. (how many periods over 2π).

5.5 Other Trig Functions

$\tan(x) = \sin(x)/\cos(x)$
$\cot(x) = \cos(x)/\sin(x)$
$\sec(x) = 1/\cos(x)$
$\csc(x) = 1/\sin(x)$