3 Solving Equations

§1 Linear Equations

Def^{n}: A linear equation (degree one) is an expression of the form $ax+b=0$.

Ex. ① Solve $4x-16=0$ for x.

$4x-16=0 \Rightarrow 4x=16 \Rightarrow x=16/4=4.$

② $\frac{2}{3}x+1=0 \Rightarrow \frac{2}{3}x=-1 \Rightarrow x=\frac{3}{2}(-1)=-3/2.$

③ $\sqrt{2}y-4+\sqrt{2}=0$

$\Rightarrow \sqrt{2}y=4-\sqrt{2}$

$\Rightarrow y=\dfrac{4-\sqrt{2}}{\sqrt{2}}=\dfrac{2^2-\sqrt{2}}{\sqrt{2}}=\dfrac{((\sqrt{2})^2)^2-\sqrt{2}}{\sqrt{2}}=\dfrac{(\sqrt{2})^4-\sqrt{2}}{\sqrt{2}}$

$=\dfrac{\sqrt{2}((\sqrt{2})^3-1)}{\sqrt{2}}=(\sqrt{2})^3-1=\sqrt{2}(\sqrt{2})^2-1=2\sqrt{2}-1.$

④ $y^2x+w^2=0$ for x.

$\Rightarrow y^2x=-w^2$

$\Rightarrow x=-w^2/y^2=-(w/y)^2.$

⑤ $4x+3=2x+1$

$\Rightarrow 4x-2x=1-3$

$\Rightarrow 2x=-2$

$\Rightarrow x=-1.$

⑥ Solve $2x+5y=3x+y+1$ for x

$\Rightarrow 5y-y-1=3x-2x$

$\Rightarrow 4y-1=x.$

⑦ Solve $2/3y+2x-1=\frac{3}{4}y+x-\frac{1}{2}$ for y

$\Rightarrow \frac{3}{4}y-2/3y=2x-1-x+1/2$

$\Rightarrow \frac{9}{12}y-\frac{8}{12}y=x-\frac{2}{2}+\frac{1}{2}=x-\frac{1}{2}$

$\Rightarrow \frac{1}{12}y=x-\frac{1}{2}$

$\Rightarrow y=12x-6.$

⑥ Shirts cost twice as much as ties
Sweaters cost twice as much as shirts.
You bought two ties, three shirts, and a sweater for \$180. How much did the sweater cost?

$s = 2t$

$w = 2s = 2(2t) = 4t$

$2t + 3s + w = 180$

$\Rightarrow 2t + 3(2t) + 4t = 2t + 6t + 4t = 12t = 180$

$\Rightarrow t = 180/12 = 15$

$\Rightarrow w = 4(15) = \$60.$

§2 Quadratic Equations

Def^n: A quadratic (degree 2) equation is one of the form

$$ax^2 + bx + c = 0$$

Quadratic Formula

Complete the square:

$$ax^2 + bx + c = 0$$

$$\Rightarrow a\left(x^2 + 2\left(\frac{b}{2a}\right) + \left(\frac{b}{2a}\right)^2\right) - a\left(\frac{b}{2a}\right)^2 + c = 0$$

$$\Rightarrow a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c = 0$$

$$\Rightarrow a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + \frac{4ac}{4a} = 0$$

$$\Rightarrow a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a}$$

$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\Rightarrow x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

E.g. ① $x^2-3x+2=0$

$\Rightarrow (x-2)(x-1)=0$

$\Rightarrow x=2$ or $x=1$

② $2x^2-3x+1=0$

$2x^2-3x+1=(2x-1)(x-1)=0$

$\Rightarrow x=\frac{1}{2}$ or $x=1$

Alternatively: $x=\dfrac{3\pm\sqrt{9-8}}{2(2)}=\dfrac{3\pm\sqrt{1}}{4}=\dfrac{3\pm1}{4}$

$\Rightarrow x=\dfrac{3+1}{4}=\dfrac{4}{4}=1$ or $x=\dfrac{3-1}{4}=\dfrac{2}{4}=\dfrac{1}{2}.$

③ $s^2+4s+4=0;\ s$

$(s+2)^2=0 \Rightarrow s=-2.$ (Repeated root: multiplicity/order 2)

④ $x^2-9=0$ or: $x^2-9=0$

$\Rightarrow (x+3)(x-3)=0$ $\Rightarrow x^2=9$

$\Rightarrow x=\pm3.$ $\Rightarrow x=\pm\sqrt{9}=\pm3.$

⑤ $y^2+2y+2=0;\ y$

$y=\dfrac{-2\pm\sqrt{4-8}}{2}=\dfrac{-2\pm\sqrt{-8}}{2}=\dfrac{-2\pm2\sqrt{-1}}{2}=-1\pm\sqrt{-1}=-1\pm i.$

This is a complex number (not a real number)

⑥ Verify this is actually a solution:

$(-1+i)^2+2(-1+i)+2=\cancel{1}-\cancel{2i}-\cancel{1}-\cancel{2}+\cancel{2i}+\cancel{2}$
$=0$

$(-1-i)^2+2(-1-i)+2=(1+i)^2-2(1+i)+2$
$=\cancel{1}+\cancel{2i}-\cancel{1}-\cancel{2}-\cancel{2i}+\cancel{2}$
$=0.$

The number b^2-4ac is called the discriminant and determines how many real roots there are:

$b^2-4ac=0 \Rightarrow$ one root of multiplicity 2,

$b^2-4ac>0 \Rightarrow$ two distinct real roots

$b^2-4ac<0 \Rightarrow$ two complex roots, no real roots.

(7) $zx+2yx^2+yz+z^2-y^2=0$; x

$$x^2(2y)+x(z+y)+(z^2-y^2)=0$$

$$x=\frac{-(z+y)\pm\sqrt{(z+y)^2-4(2y)(z^2-y^2)}}{4y}.$$

§3 Other Types of Equations

Eg: (1) $\sqrt{x}-3=0$; x

$\Rightarrow \sqrt{x}=3$

$\Rightarrow x=9$

(2) $(\sqrt{x}+2)^3-64=0$

$\Rightarrow (\sqrt{x}+2)^3=64$

$\Rightarrow \sqrt{x}+2=\sqrt[3]{64}=4$

$\Rightarrow \sqrt{x}=4-2=2$

$\Rightarrow x=4.$

(3) $\dfrac{1}{x-5}+\dfrac{1}{x+5}=\dfrac{10}{x^2-25}$

$\dfrac{10}{x^2-25}=\dfrac{10}{(x+5)(x-5)} \Rightarrow (x+5)+(x-5)=10$

$\Rightarrow 2x=10$

$\Rightarrow x=5$

But this is absurd! No solutions

④ $x^4-5x^2+4=0$

$X=x^2 \Rightarrow X^2-5X+4=0$

$\Rightarrow (X-4)(X-1)=0$

$\Rightarrow X=4$ or $X=1$

$\Rightarrow x^2=4$ or $x^2=1$

$\Rightarrow x=\pm 2$ or $x=\pm 1$

$\Rightarrow x^4-5x^2+4=(x+2)(x-2)(x+1)(x-1)$.

⑤ $x^4-5x^2-36=0$

$X=x^2 \Rightarrow X^2-5X-36=0$

$\Rightarrow (X-9)(X+4)=0$

$\Rightarrow X=9$ or $X=-4$

$\Rightarrow x^2=9$ or $x^2=-4$

$\Rightarrow x^2=\pm 3$ are the only real roots.

There are two complex solutions, $x=\pm i4$.

⑥ $x^6+6x^3-16=0$

Let $X=x^3 \Rightarrow X^2+6X-16=0$

$\Rightarrow (X+8)(X-2)=0$

$\Rightarrow X=-8$ or $X=2$

$\Rightarrow x^3=-8$ or $x^3=2$

$\Rightarrow x=-2$ or $x=\sqrt[3]{2}$

⑦ $x+\sqrt{x}-6=0$

Let $X=\sqrt{x}$

$\Rightarrow X^2+X-6=0$

$\Rightarrow (X+3)(X-2)=0$

$\Rightarrow X=-3$ or $X=2$

$\Rightarrow \sqrt{x}=-3$ or $\sqrt{x}=2$

$\Rightarrow x=4$

$\sqrt{x}=-3$ is impossible!

⑨ $(x^2-4)(x^2+2x-3)=0$

$\Rightarrow (x+2)(x-2)(x+3)(x-1)=0$

$\Rightarrow x \in \{\pm 2, 1, -3\}$.

⑩ $x^5+3x^4-4x^3=0$

$\Rightarrow x^3(x^2+3x-4)=0$

$\Rightarrow x^3(x+4)(x-1)=0$

$\Rightarrow x \in \{0, 1, -4\}$ (0 is a root of multiplicity 3)