2. Completing the Square

Let $f(x) = ax^2 + bx + c$. Factor out the leading coefficient

$$f(x) = a\left(x^2 + \frac{b}{a}x\right) + c = a\left(x^2 + 2\left(\frac{b}{2a}\right)x\right) + c$$

then add and subtract $\left(\frac{b}{2a}\right)^2$ to get

$$\begin{aligned} f(x) &= a\left(x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c \\ &= a\left(x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2\right) - a\left(\frac{b}{2a}\right)^2 + c \\ &= a\left(x + \left(\frac{b}{2a}\right)\right)^2 - \frac{b^2}{4a} + c \\ &= a\left(x + \left(\frac{b}{2a}\right)\right)^2 - \frac{b^2 - 4ac}{4a} \end{aligned}$$

Eg. ① $f(x) = x^2 + 8x + 12 = x^2 + 2(4)x + 12$

$$\begin{aligned} &= (x^2 + 2(4)x + 16) - 16 + 12 \\ &= (x+4)^2 - 4 \end{aligned}$$

② $f(x) = x^2 - 3x + 4 = \left(x^2 - 2\left(\frac{3}{2}\right)x + \frac{9}{4}\right) - 9/4 + 4$

$$\begin{aligned} &= (x - 3/2)^2 - 9/4 + 16/4 \\ &= (x - 3/2)^2 + 7/4 \end{aligned}$$

③ $f(x) = 4x^2 + 20x - 100 = 4\left(x^2 + 2\left(\frac{5}{2}\right)x\right) - 100$

$$\begin{aligned} &= 4\left(x^2 + 2(5/2)x + \frac{25}{4} - 25/4\right) - 100 \\ &= 4\left(x^2 + 2(5/2)x + \frac{25}{4}\right) - 25 - 100 \\ &= 4(x + 5/2)^2 - 125 \end{aligned}$$

④ $x^2 - 4x + y^2 + 6y = 2$

$\Rightarrow (x^2 - 2(2)x + 4) + (y^2 + 2(3)y + 9) = 2 + 4 + 9 = 15$

$\Rightarrow (x-2)^2 + (y+3)^2 = 15.$

Rmk: This is the equation of a circle of radius $\sqrt{15}$ centered at $(2,-3)$.

⑤ $4x^2-9y^2+8x+18y-25=0$

$\Rightarrow 4(x^2+2x)-9(y^2-2y)=25$

$\Rightarrow 4(x^2+2x+1)-9(y^2-2y+1)=25+4-9$

$\Rightarrow 4(x+1)^2-9(y-1)^2=20$

⑥ $x^2-\pi x+2y^2-y=0$

$\Rightarrow \left(x^2-2(\frac{\pi}{2})x+(\pi/2)^2\right)+2\left(y^2-2(\frac{1}{4})y+\frac{1}{16}\right)=(\pi/2)^2+\frac{1}{8}$

$\Rightarrow (x-\pi/2)^2+2(y-\frac{1}{4})^2=(\pi/2)^2+1/8=\frac{2\pi^2+1}{8}$

⑦ Determine whether $x^2+y^2-4x-6y-3=0$ is the equation of a circle $((x-h)^2+(x-k)^2=r^2)$. If so, find the radius and center.

$x^2+y^2-4x-6y-3=(x^2-2(2)x)+(y^2-2(3)y)-3=0$

$\Rightarrow (x^2-2(2)x+4)+(y^2-2(3)y+9)=3+4+9=16$

$\Rightarrow (x-2)^2+(y-3)^2=4^2$

So yes; radius is 4, center is $(2,3)$.

⑧ Find the radius and center of the circle given by $2x^2+2y^2-4x+12y=-10$

$2x^2+2y^2-4x+12y=-10 \Rightarrow x^2+y^2-2x+6y=-5$

$\Rightarrow (x^2-2x+1)+(y^2-2(3)y+9)=-5+1+9=\cancel{9}\,5$

$\Rightarrow (x-1)^2+(y-3)^2=5$

radius $r=\sqrt{5}$, centered at $(1,3)$.