1: Arithmetic

Multiplying and Dividing Fractions

Def^{n}. $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}$

Eg. ① $\dfrac{2}{3} \cdot \dfrac{5}{7} = \dfrac{10}{21}$ ② $\dfrac{1}{9}\left(\dfrac{-5}{8}\right) = \dfrac{-5}{72}$

③ $\left(\dfrac{1}{4}\right)\left(\dfrac{7}{5}\right)\left(\dfrac{3}{8}\right) = \dfrac{21}{160}$ ④ $\left(\dfrac{-1}{7}\right)\left(\dfrac{3}{8}\right)\left(\dfrac{-2}{\pi}\right) = \left(\dfrac{-1}{7}\right)\left(\dfrac{3}{4}\right)\left(\dfrac{-1}{\pi}\right) = \dfrac{3}{28\pi}$.

⑤ Reduce 30/84

$\dfrac{12}{7}$ $\dfrac{30}{84} = \dfrac{2 \cdot 3 \cdot 5}{2^2 \cdot 3 \cdot 7} = \dfrac{5}{2 \cdot 7} = \dfrac{5}{14}$.

Don't cancel across sums. $(3+x^2)/3 \neq x^2$.

However
$$\frac{3+3x^2}{3} = \frac{3(1+x^2)}{3} = 1+x^2.$$

Def^{n}. $\dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{a}{b} \cdot \dfrac{d}{c} = \dfrac{ad}{bc}$.

Eg. ① $(-1/3)/(5/6) = \left(-\dfrac{1}{3}\right)\left(\dfrac{6}{5}\right) = (-1)(2/5) = \boxed{-2/5}$.

② $(2/3)/(3/8) = \dfrac{2}{3} \cdot \dfrac{8}{3} = \boxed{\dfrac{16}{9}}$

③ $(5/4)/(-10/3) = (5/4)(3/-10) = -\left(\dfrac{1}{4}\right)\left(\dfrac{3}{2}\right) = \boxed{3/8}$

④ Simplify $\left(\dfrac{x+1}{y}\right) \cdot \left(\dfrac{-y+1}{x}\right)$

$= \dfrac{-(xy - x + y - 1)}{xy} = \dfrac{1 - xy + x - y}{xy}$.

⑤ $\dfrac{\left(\dfrac{x^2y}{z}\right)}{\left(\dfrac{xy^2}{z^3}\right)}$

$= \left(\dfrac{x^2y}{z}\right)\left(\dfrac{z^3}{xy^2}\right) = \boxed{\dfrac{xz^2}{y}}$

Adding and Subtracting Fractions

Def^n: $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$. Here, bd is the common denominator.

Rmk: This always works, but the least common multiple is usualy more efficient.

E.g. ① $\frac{3}{5} + \frac{1}{2} - \frac{2}{3} = \frac{(2)(3)(3)+(3)(5)-(2)(5)(2)}{(2)(3)(5)}$

$= \frac{18+15-20}{30} = \boxed{\frac{13}{30}}$

b) $\frac{1}{6} - \frac{1}{9} = \frac{1}{(2)(3)} - \frac{1}{(3)(3)} = \frac{3-2}{(2)(3)(3)} = \boxed{\frac{1}{18}}$

c) $\dfrac{\frac{1}{2}+\frac{3}{4}}{\frac{1}{3}-\frac{1}{6}} = \dfrac{\frac{2+3}{4}}{\frac{2-1}{6}} = \left(\frac{5}{4}\right)\left(\frac{6}{1}\right) = \frac{5(3)}{2} = \boxed{\frac{15}{2}}$

E.g. ② $\frac{xy}{z} + \frac{x}{5} = \boxed{\frac{5xy+zx}{5z}}$

③ $\left(\frac{3a+2b}{5ab}\right) \Big/ \left(\frac{a}{b} - \frac{b}{a}\right) = \left(\frac{3a+2b}{5ab}\right)\left(\frac{ab}{a^2-b^2}\right) = \frac{3a+2b}{5(a^2-b^2)}$

Parentheses

Ex: $5-(1+\frac{1}{2})+(3-4)-(7-\frac{1}{2}) = 5-1-\frac{1}{2}+3-4-7+\frac{1}{2}$

$= 8-7-5$

$= \boxed{-4}$

E.g. $3-2(8+1)-3(5-7) = 3-16-2-15+21$

$= 24-33$

$= \boxed{-9}$

E.g. $xy-(2x-y)-2y(1-x) = xy-2x+y-2y+2xy$

$= \boxed{3xy-2x-y}$

E.g. $3x^2y-(x^2-y^3)-2y(x-y) = 3x^2y-x^2+y^3-2xy+2y^2$

Exponents

Laws

1) $a^m a^n = a^{m+n}$,
2) $a^m / a^n = a^{m-n}$; $a \neq 0$,
3) $(a^m)^n = a^{mn}$,
4) $(ab)^n = a^n b^n$,
5) $\left(\frac{a}{b}\right)^n = a^n / b^n$; $b \neq 0$,
6) $\left(\frac{a}{b}\right)^{-n} = a^{-n} / b^{-n} = \frac{b^n}{a^n} = \left(\frac{b}{a}\right)^n$; $a, b \neq 0$.

E.g.

1) a) $\frac{4^2-1}{3^3-2^2} = \frac{16-1}{27-4} = \frac{15}{23}$.

b) $5^{-1} + 3^{-1} = \frac{1}{5} + \frac{1}{3} = \frac{3+5}{15} = \frac{8}{15}$.

c) $\frac{3 \cdot 8^2}{9 \cdot 8^3} = \left(\frac{3}{9}\right)\frac{1}{8}\left(\frac{8}{8}\right)^2 = \frac{3}{(9)8} = \frac{3}{72}$.

2) $\frac{x^2 y^5}{x^{-3}} \div \frac{x^{-5} y^4}{x^3} = (x^2)(x^3) y^5 \div \frac{y^4}{(x^5)(x^3)} = x^5 y^5 \div \frac{y^4}{x^8}$

$= \frac{(x^5) x^8 y^5}{y^4} = x^{13} y$.

3) $\frac{1}{a^3} - \left(\frac{1}{a^5} - \frac{1}{a^2}\right) = \frac{1}{a^3} - \left(\frac{1-a^3}{a^5}\right) = \frac{a^2 - (1-a^3)}{a^5}$

$= \frac{a^3 + a^2 - 1}{a^5}$.

4) $\left(\frac{x^{-2}}{x^8}\right)^{-2} = \left(\frac{1}{(x^2)(x^8)}\right)^{-2} = \left(\frac{1}{x^{10}}\right)^{-2} = (x^{10})^2 = x^{20}$.

5) a) $(x^2 y^3)^5 = x^{10} y^{15}$

b) $\left(\frac{a^{-3}}{b}\right)^4 = \left(\frac{1}{a^3 b}\right)^4 = \frac{1}{a^{12} b^4}$.

c) $\left(\frac{x^5}{y^3}\right)^{-2} = \left(\frac{y^3}{x^5}\right)^2 = \frac{y^6}{x^{10}}$.

⑥ Can we simplify $(x^2+y^3)^7$?

<u>Rmk</u>: $(x^2+y^3)^7 \neq x^{14}+y^{21}$!

$$
\begin{array}{c|ccccccccc}
 & & & & 1 & & & \\
1 & & & 1 & & 1 & & \\
2 & & & 1 & 2 & 1 & & \\
3 & & 1 & 3 & & 3 & 1 & \\
4 & & 1 & 4 & 6 & 4 & 1 & \\
5 & 1 & 5 & 10 & & 10 & 5 & 1 \\
6 & 1 & 6 & 15 & 20 & 15 & 6 & 1 \\
7 & 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1
\end{array}
$$

$x^3+3x^2y+3xy^2+y^3$

x^4

$$(x^2+y^3)^7 = (x^2)^7 + 7(x^2)^6y^3 + 21(x^2)^5(y^3)^2 + 35(x^2)^4(y^3)^3 + 35(x^2)^3(y^3)^4 + 21(x^2)^2(y^3)^5 + 7(x^2)(y^3)^6 + (y^3)^7$$
$$= x^{14} + 7x^{12}y^3 + 21x^{10}y^6 + 35x^8y^9 + 35x^6y^{12} + 21x^4y^{15} + 7x^2y^{18} + y^{21}.$$

<u>Binomial Theorem</u>

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$; show

$\binom{n}{k} = \frac{n!}{(n-k)!k!}$ is the k^{th} entry (left to right) of the n^{th} row of Pascal's Triangle.

⑦ a) $(x^2y^2)^{10} = x^{20}y^{20}$

<u>Roots</u>

Def^n: Say a number y is an n^{th} root of the number x if $y^n = x$. The number n is the order of the root.

Suppose that $n=2$, and $y^2 = x$ (that is y is the square root of x). We know $(-1)^2=1$, so it follows from
$$(-y)^2 = (-1)^2y^2 = (1)x = x$$
that $-y$ is also a square root of x. This is true in general:

If n is even, then $2 \mid n$, say $n = 2m$ for some $m \in \mathbb{Z}$. Then if $y^n = x$, we have

$$(-y)^n = (-1)^n y^n = (-1)^{2m} x = ((-1)^2)^m x = 1^m x = x,$$

so $-y$ is also an n^{th} root of x.

Note this is not true if n is odd. In this case, $n = 2m+1$, so

$$(-1)^n = (-1)^{2m+1} = (-1)(-1)^{2m} = (-1)(1) = -1$$

In particular, this means the n^{th} roots of negative numbers exist when n is odd, but not when n is even. (as real numbers).

Notation: If $y^n = x$, then we write $y = \sqrt[n]{x}$. If n is even, the symbol $\sqrt[n]{x}$ refers to the positive n^{th} root.

Eg. $\sqrt[4]{16} = 2$, $-\sqrt[4]{16} = -2$

$(2)^4 = (-2)^4 = 16$.

$\sqrt[3]{27} = 3$, $\sqrt[3]{-27} = -3$

$3^3 = 3 \cdot 3^2 = 3 \cdot 9 = 27$

$(-3)^3 = (-1)^3 \cdot 3^3 = (-1)27 = -27$.

Fractional Exponents

Def^n: For n an integer, we define

$$x^{1/n} = \sqrt[n]{x}.$$

For another integer m,

$$x^{m/n} = (\sqrt[n]{x})^m = \sqrt[n]{x^m},$$

which follows from

$$x^{m/n} = (x^{1/n})^m = (x^m)^{1/n}.$$

Eg: ① a) $8^{2/3} = (\sqrt[3]{8})^2 = (2)^2 = 4$.

b) $\left(\frac{-1}{27}\right)^{4/3} = \left(\sqrt[3]{\frac{-1}{27}}\right)^4 = \left(\frac{\sqrt[3]{-1}}{\sqrt[3]{27}}\right)^4 = \left(\frac{-1}{3}\right)^4 = \frac{1}{81}$

c) $(-32)^{4/5} = (\sqrt[5]{-32})^4 = (-2)^4 = 16.$

Laws

Let r, s be rational numbers. Then the rules are identical to those for integers

Rmk: Again, $\sqrt{a+b} = (a+b)^{1/2} \neq \sqrt{a} + \sqrt{b}$. There is no convenient binomial analog

Percentages

Def^n: By % we mean "per one-hundred". In general, $x\%$ of y is

$$\left(\frac{x}{100}\right) y.$$

Ex: ① 15% of 90

$$\frac{15}{100} \cdot 90 = \frac{15 \cdot 9}{10} = \frac{3 \cdot 9}{2} = \frac{27}{2}$$

② 1% of 320

$$\frac{1}{100} \cdot 320 = \frac{32}{10}$$

③ $\frac{1}{3}$% of 930

$$\frac{1}{300} \cdot 930 = \frac{93}{30} = \frac{31}{10}$$

$\frac{2 \cdot 20}{100} = \frac{2}{5}$

④ 250% of 150

$$\frac{250}{100} \cdot 150 = 25 \cdot 15 = 375$$

⑤ A CD usually sells for \$15.99. A record store is having a 60% off sale. How may CDs can you get for \$20?

Each CD costs

$$15.99 \left(\frac{40}{100}\right) = \frac{2}{5}(15.99) = 6.40 \text{ (rounded up)},$$

so $20/(6.40) = 3.125$, so you can buy 3 CDs at a cost of \$19.20.

⑥ In 1995, the federal budget was cut by 30% to 147 billion. What whas the budget before it was cut?

Let B be the current budget. Then

$$\frac{70}{100}B = \frac{7}{10}B = 147$$

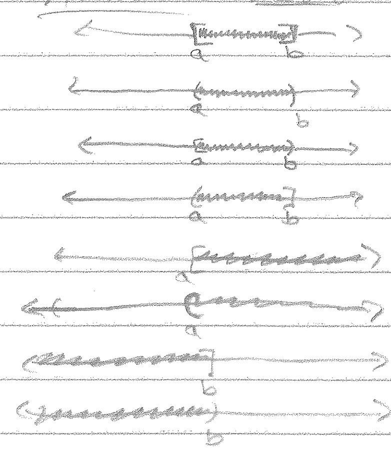
so

$$B = 147\left(\frac{10}{7}\right) = 21(10) = 210 \text{ billion dollars.}$$

Intervals

An interval is a connected piece of the real number line.

Interval Notation	Number Line	Inequalities
$[a,b]$		$a \le x \le b$,
(a,b)		$a < x < b$,
$[a,b)$		$a \le x < b$,
$(a,b]$		$a < x \le b$.
$[a,\infty)$		$a \le x$
(a,∞)		$a < x$
$(-\infty,b]$		$x \le b$
$(-\infty,b)$		$x < b$



Def^n Let A and B be two sets.

a) $A \cap B = \{x \in A \text{ and } x \in B\}$

b) $A \cup B = \{x \in A \text{ or } x \in B\}$

c) The empty set is $\emptyset$.

E.g. ① a) $(-\infty, 5) \cap (3, \infty) = (3, 5)$,

b) $(-\infty, 10] \cap (-1, \infty) = (-1, 10]$,

c) $(-\infty, 10) \cap (21, \infty) = \emptyset$

d) $(-\infty, 5) \cup (4, \infty) = (-\infty, \infty) = \mathbb{R}$

e) $(-\infty, 5) \cup (10, \infty)$ is reduced (disconnected).

f) $(-\infty, 5) \cap [5, 10] = \emptyset$.