3.2 Exponential Models: Comparing Rates

Changing Time Periods

Suppose a researcher measures the growth of an insect population and wants to know the weekly growth factor. Given a daily growth factor of a, the population undergoes 7 daily time periods over a week, so the weekly growth factor is a^7. Similarly, if b is the weekly growth factor, then the daily growth factor, a, must satisfy $a^7 = b$. In particular, the daily growth factor is $a = b^{1/7}$.

Growth Factor	
Daily: a	weekly: a^7
Weekly: b	Daily: $b^{1/7}$
Monthly: c	Yearly: c^{12}
Yearly: d	Monthly: $d^{1/12}$

Eg.: A biologist finds the 30 minute growth rate ~~factor~~ for a bacteria pop. is 0.85.
Find the one hour growth rate

$\underline{a = 1+r} = 1 + 0.85 = 1.85.$ – 30 min. growth factor

$a^2 = (1.85)^2 \approx 3.4$ – 60 min growth factor

$r = a-1 \approx 3.4 - 1 = 2.4.$

hourly growth rate

hourly growth factor

Eg.: A chinchilla farm starts with 20 chinchillas, after 3 years there are 128.
Assume the growth is exponential.

3 year growth factor is $\frac{128}{20} = 6.4 = b$

Find the 1-year growth factor

Let a be the one-year growth factor. Since three years, we know

$a^3 = b = 6.4$

$\Rightarrow a = \sqrt[3]{6.4} \approx 1.86.$

Growth of an Investment: Compound Interest

Suppose \$1000 is invested in a 16-year certificate of Deposit (CD), paying 6% interest annually and compounded monthly.

This means that each month, 6%/12 of the account balance is added to the account.

Start with \$1000 at month 0.

Month 1: $\underline{1000 + \frac{6}{100}\left(\frac{1}{12}\right)1000}$

Month 2: $\underbrace{\left(1000 + \frac{6}{100}\left(\frac{1}{12}\right)1000\right)}_{\text{previous balance}} + \frac{6}{100}\left(\frac{1}{12}\right)\underbrace{\left(1000 + \frac{6}{100}\left(\frac{1}{12}\right)1000\right)}_{\text{previous balance}}$

$\vdots$

Month 1: $1000\left(1 + \frac{1}{200}\right) = 1000(1 + .005)$

$= \cancel{1000(1+} \ \underline{1000(1.005)}^{1}$

Month 2: $\underline{1000(1.005)} + (.005)\left(\underline{1000(1.005)}\right)$

$= 1000(1.005)(1 + .005)$

$= 1000(1.005)^2$.

Month 3: $1000(1.005)^3$

P - principal

r - interest

n - compounding period

t - number of years

The amount of the investment after t years is

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

E.g.: \$5000 in a 3-year CD

Either A: 5.50% each year compounded twice a year, or

B: 5.50% compounded daily.

Which is the better investment?

$$A_1(t) = 5000\left(1 + \frac{5.50}{100}\left(\frac{1}{2}\right)\right)^{2\cdot 3} = \$5883.84$$

$$A_2(t) = 5000\left(1 + \frac{5.50}{100}\left(\frac{1}{365}\right)\right)^{365\cdot 3} = \$5896.89.$$

Annual Percentage Yield (APY)

Find the APY for cert. B: $A_2(t) = 5000\left(1 + \frac{0.055}{365}\right)^{365\cdot 1}$

$$= 5000\,\underbrace{(1.0565)}_{\text{annual growth factor}}$$

$APY = 1.0565 - 1 = .0565 \approx 5.65\%$.

3.3 Comparing Linear and Exponential Growth

Let $f(x) = Ca^x$ exponential growth (or decay) model. The average rate of change of $f(x)$ over one time period is

$$\frac{f(x+1) - f(x)}{(x+1) - x} = f(x+1) - f(x).$$

$$= Ca^{x+1} - Ca^x$$

As a proportion of $f(x)$, this is just the rate (growth or decay). This is

$$\frac{f(x+1) - f(x)}{f(x)} \quad (= r)$$

is the percentage rate of change

Eg: Find the average rate of change and percentage rate of change for $f(x) = 10 \cdot 3^x$ on intervals of length 1 starting at 0.

Percentage rate of change is

growth rate: $r = a - 1 = 3 - 1 = 2$.

percentage rate of change: 200% – this is constant.

True for any exponential function.