E.g.: $\log_4(8)$

Recall: $\log_b(x) = \dfrac{\log_a(x)}{\log_a(b)}$

$$\log_4(8) = \frac{\log_2(8)}{\log_2(4)} = \frac{\log_2(2^3)}{\log_2(2^2)} = \frac{3}{2}$$

E.g.: $\log_9(27) = \dfrac{\log_3(27)}{\log_3(9)} = \dfrac{\log_3(3^3)}{\log_3(3^2)} = \dfrac{3}{2}$.

4.4: The natural exponential and logarithm

The number $e \approx 2.718\ldots$ is the limit of $\left(1+\frac{1}{n}\right)^n$ as n tends to infinity

$$\lim_{n \to \infty} \left(1+\frac{1}{n}\right)^n = e.$$

The natural exponential function is

$$f(x) = e^x$$

The natural logarithm is the inverse of this function

$$\underline{\ln(x)} := \log_e(x).$$

Recall by change of base, with base a,

$$\log_a(x) = \frac{\ln(x)}{\ln(a)} = \left(\frac{1}{\ln(a)}\right)\ln(x).$$

$$\ln(1) = 0$$

$$\ln(e) = 1$$

$$\left.\begin{array}{l} \ln(e^x) = x \\ e^{\ln(x)} = x \end{array}\right\} \underline{\text{Inverse Functions}}$$

$$\ln(xy) = \ln(x) + \ln(y)$$

$$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$$

$$\ln(x^c) = c\ln(x).$$

<u>Recall</u> Compounding Interest:

$$A(t) = P\left(1+\frac{r}{n}\right)^{nt}$$

r - rate, n - # of compounding periods,

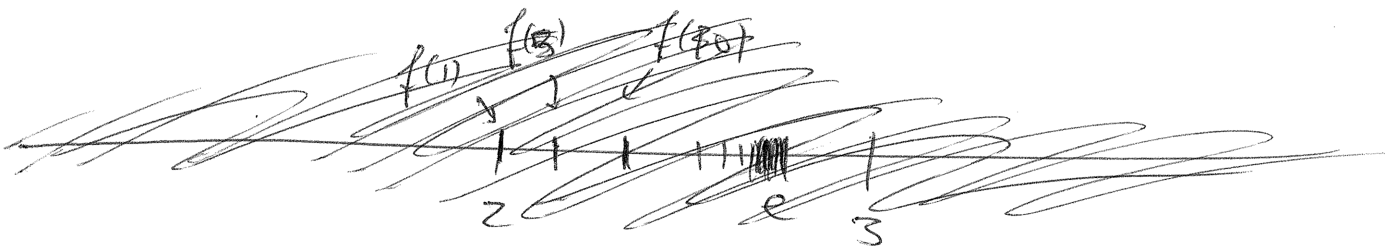
P principal, t - time (years)

Let $m = n/r$, then $\frac{1}{m} = \frac{r}{n}$, $nt = \frac{n}{r}(rt)$

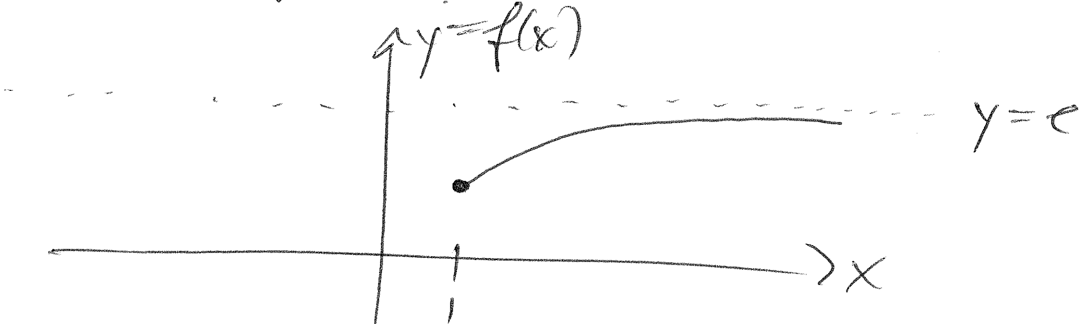
$$A(t) = P\left(\left(1+\frac{1}{m}\right)^{\frac{n}{r}}\right)^{rt}$$

$$= P\left(\left(1+\frac{1}{m}\right)^{m}\right)^{rt}$$

$$e = \lim_{m \to \infty}\left(1+\frac{1}{m}\right)^{m}, \quad f(x) = \left(1+\frac{1}{x}\right)^{x}$$



Graph $f(x) = \left(1+\frac{1}{x}\right)^{x}$ for $x \geq 1$



The line $y=e$ is a horizontal asymptote for $f(x)$.

$A(t) = P\left(\left(1+\frac{1}{m}\right)^m\right)^{rt}$ as the # of compounding periods increases,

$$\left(1+\frac{1}{m}\right)^m$$

approaches the value e. So if we compound instantaneously (every instant, the interest compounds), we can say that the value of the account after t years is

$$\underline{A(t) = Pe^{rt}.}$$

This is called continuously compounding interest.

<u>Exponential Growth or Decay</u>

$$f(t) = Ce^{rt}$$

models growth if $r>0$ and decay if $r<0$.

C - initial value

t - time

r - instantaneous growth rate.

E.g.: Population: 500 CFU/mL

Instantaneous growth rate: 40%/hour

$C = 500$

$r = 40\% = 4/10 = .4$

$P(t) = 500e^{.4t}$

$P(7) = 500\, e^{.4 \cdot 7} \approx 27{,}300$ CFU/mL

represents the size of the sample after 7 hours.

E.g.: $f(x) = 350(1.4)^x$. We would like to change the base of this exponential function to e.

Recall: $e^{\ln(x)} = x$

$\Rightarrow e^{\ln(1.4)} = 1.4$

$\Rightarrow f(x) = 350\,(1.4)^x = 350\left(e^{\overset{r}{\overbrace{\ln(1.4)}}}\right)^x$

$= 350e^{(\ln(1.4))x}$