Which Functions have Inverses?

Recall that a function f has an inverse, if there exists a function, f^{-1}, such that for every x in the domain of f,

$$f^{-1} \circ f(x) = x$$

and for every y in the range of f,

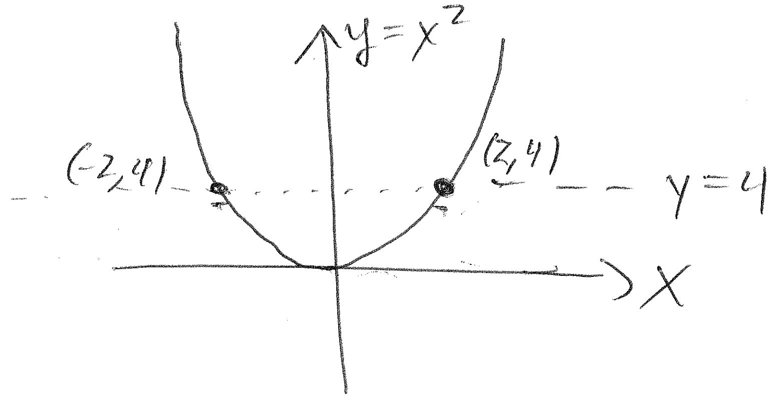
$$f \circ f^{-1}(y) = y.$$

Eg: Not every function has an inverse. Take the function $f(x) = x^2$, defined on all the real numbers.

Recall: $y = x^2$ determines y as a function of x but does not determine x as a function of y because if we solve for $\cancel{x}$, then

$$x = \pm \sqrt{\cancel{x}}$$

The choice of value for x: either $\sqrt{y}$ or $-\sqrt{y}$ says this is not a function.



To be an inverse, we would need to have a function which takes in the y-value of a coordinate pair (x,y) on the graph of $f(x)$ and outputs the x-value.

If the graph of my function f intersects a horizontal line in two points, say (x_1,y) and (x_2,y), then we know $f(x_1)=f(x_2)=y$.
So there cannot be an inverse function because any rule assigning an x value to the value y involves a choice:

Either $f^{-1}(y)=x_1$ or $f^{-1}(y)=x_2$.

So we at least must require that the any horizontal line intersects the graph of f in at most one point.

This is also sufficient: if any horizontal line intersects the graph in at most one point, then we can define an inverse function. If (x,y) is any coordinate pair on the graph of f, then $y=f(x)$ ③

and we define $f^{-1}(y)=x$.

Def^{n}: We say a function f, is one-to-one if for every value of x and x' in the domain of f, $f(x) \neq f(x')$, when $x \neq x'$.

<u>Horizontal Line Test</u>

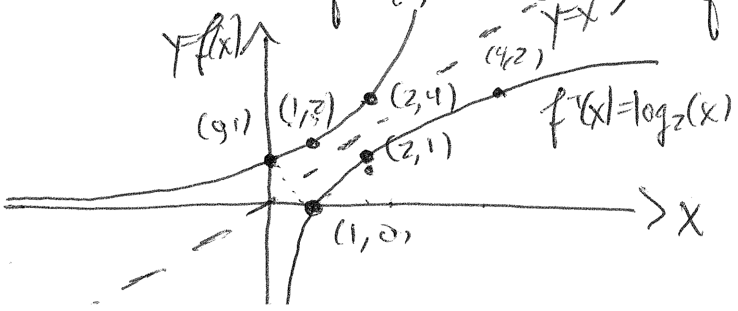
If any horizontal line passes through at most one point on the graph of f, then f is one-to-one.

Every one-to-one function has an inverse.

<u>4.1</u>

<u>Graphs of logarithms</u>

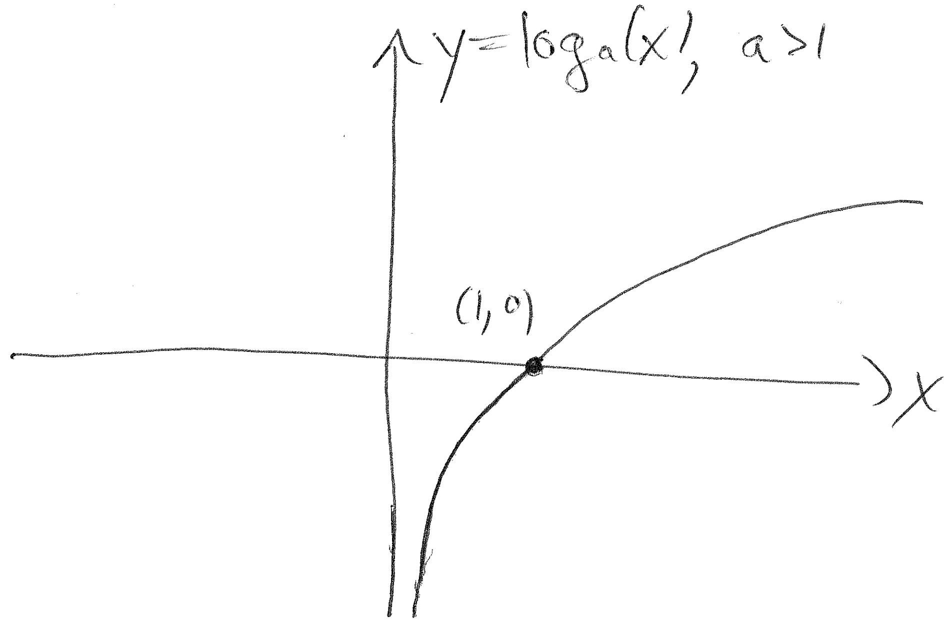
Consider $f(x)=2^x$, $f^{-1}(x)=\log_2(x)$



Know the domain of f^{-1} is the range of f (positive y-axis) and $f^{-1}\circ f(x)=x$

The graph of $\log_2(x)$ is a reflection ④
of the graph of 2^x across the line
$y=x$.

When $a>1$, the graph of $\log_a(x)$ is the reflection of a^x across $y=x$ and looks like



<u>4.2</u> Laws of Logarithms

1. $\log_a(AB) = \log_a(A) + \log_a(B)$
2. $\log_a(A/B) = \log_a(A) - \log_a(B)$
3. $\log_a(A^c) = c\log_a(A)$

E.g.: a) $\log_4(2) + \log_4(32) = \log_4(2 \cdot 32)$ 5

$$= \log_4(64)$$
$$= \log_4(4^3)$$
$$= 3.$$

b) $\log_2(80) - \log_2(5) = \log_2\left(\frac{80}{5}\right) = \log_2(16) = \log_2(2^4) = 4.$

c) $-\frac{1}{3}\log_2(8) = \log_2\left(8^{-1/3}\right) = \log_2\left(\frac{1}{\sqrt[3]{8}}\right) = \log_2\left(\frac{1}{2}\right)$

$$= \log_2(2^{-1})$$
$$= -1.$$