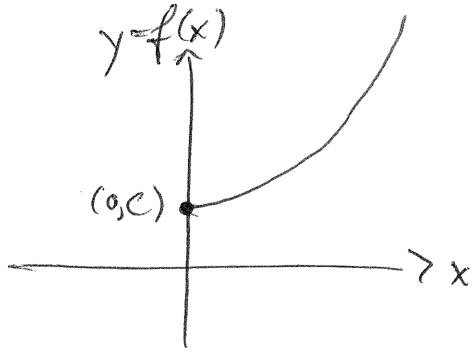
3.1: Exponential Growth and Decay

Exponential Growth is modeled by a function of the form

$$f(x) = Ca^x, \quad a > 1, \quad C \text{ is a constant,}$$
(a also a constant).

- The variable x is the number of time periods,
- The base, a, is the growth factor, the factor by which $f(x)$ is multiplied after one time period,
- The constant C is the initial value of f, i.e. when $x=0$, $f(0) = Ca^0 = C$.
- The graph of such a function has the shape



E.g.: A bacterial infection starts with 100 bacteria and the population triples every hour

hours → x	# bacteria
0	$100 = 3^0 \cdot 100$
1	$300 = 3^1 \cdot 100$
2	$900 = 3^2 \cdot 100$
3	$2700 = 3^3 \cdot 100$
⋮	

$$f(x) = 100 \cdot 3^x$$

This is an exponential growth model for the bacteria population

Eg: A ~~frog~~ pond is stocked with 5800 fish, and each year the population is multiplied by a factor of 1.2.

$$f(x) = 5800(1.2)^x$$

Suppose a population is modeled by

$$f(x) = Ca^x.$$

Increasing x by one time period, we have

$$f(x+1) = Ca^{(x+1)} = \underbrace{Ca^x}_{f(x)} a = a f(x)$$

This gives the growth factor, a, as

$$a = \frac{f(x+1)}{f(x)}.$$

Eg: A chinchilla farm starts with 20 chinchillas, after 3 years there are 128 chinchillas. Assume the number of chinchillas grows exponentially. Find the 3-year growth factor.

$$\frac{f(3)}{f(0)} = \frac{128}{20} = 6.4.$$

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The growth rate of a population is the proportion of the population by which it increases during one time period. This is,

$$r = \frac{f(x+1) - f(x)}{f(x)}$$

For example, if a population of 20 chinchillas increases to 25 over one year, then the growth rate is

$$r = \frac{25-20}{20} = \frac{5}{20} = 25\%$$

Given a exponential growth model $f(x) = Ca^x$, then

$$r = \frac{f(x+1) - f(x)}{f(x)} = \frac{Ca^{x+1} - Ca^x}{Ca^x}$$

$$= \frac{\cancel{Ca^x} \cdot a}{\cancel{Ca^x}} - \frac{\cancel{Ca^x} \cdot 1}{\cancel{Ca^x}}$$

$$= a - 1$$

So $r = a-1$ and $a = r+1$.

Recall that for exponential growth, $f(x) = Ca^x$,

$a > 1$, and so

$$r = a - 1 > 0$$

E.g.: 50 rabbits, population grows exponentially, increasing by 60% each year.

<u>Note</u>: This is <u>not</u> a growth factor.

From year 0 to year 1, the population increases by 60% of 50, which is

$$50 \cdot \frac{6}{10} = 5 \cdot 6 = 30$$

and from year 1 to year 2, the population increases by

$$\frac{6}{10}(50 + 30) = \frac{6}{10}(80) = 6 \cdot 8 = 48.$$

The growth factor is given by

$$a = 1 + r = 1 + \left(\frac{6}{10}\right) = 1.6$$

So the model for this exponential growth is

$$f(x) = 50(1.6)^x.$$

b) How many rabbits after 8 years?

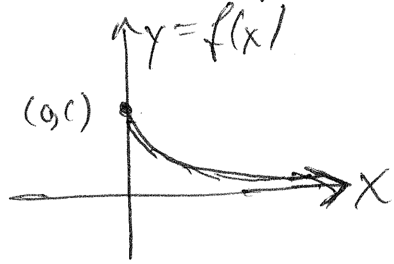
$f(8) = 50(1.6)^8 \simeq 2147.48$, so approximately

2147 rabbits

Exponential ~~Decay~~ Decay

Exponential Decay is modeled by a function of the form $f(x) = Ca^x$, $0 < a < 1$.

- The variable x is the number of time periods,
- a is called the decay factor,
- The decay rate satisfies $a = 1 + r$, so the decay rate is <u>negative</u>
- The graph of this type of function is



Eg.: A patient is administered 75 mg of a therapeutic drug. It is known that 30% of the drug is expelled from the body each hour.

a) Find an exponential decay model for the amount of the drug remaining after x hours.

$$a = 1 + r = 1 + (-3/10) = 7/10 = .7$$

So

$$f(x) = 75(.7)^x$$

b) How much remains after 4 hours?

$$f(x) = 75(.7)^4 \approx 18.008 \text{ mg.}$$

Approximately 18 mg remain after 4 hours.

E.g.: The half-life of radium-226 is 1600 years. A 50-gram sample of radium-226 is placed in an underground facility & monitored.

a) Find a function m(x) modeling the mass of the sample after x half-lives.

$$f(x) = \cancel{\$}\, 50\left(\tfrac{1}{2}\right)^x$$

b) How much remains after 4000 years?

$$\frac{4000 \text{ years}}{1600 \text{ years/half-life}} = 2.5$$

$$m(2.5) = 50\left(\tfrac{1}{2}\right)^{2.5} \approx 8.84 \text{ g.}$$