4.6 Working with Functions: Composition & Inverse

Def^n: Given two functions f and g, we can make a new function called the composition of f with g

$$(f \circ g)(x) = f(g(x))$$

provided the range of g is contained in the domain of f.

E.g.: $f(x) = x+2$, $g(x) = 2x$

$$f \circ g(x) = f(g(x)) = f(2x) = 2x+2.$$

E.g.: f, g as above

$$(g \circ f)(x) = g(f(x)) = g(x+2) = 2(x+2) = 2x+4$$

$$\cancel{=4x+4}$$

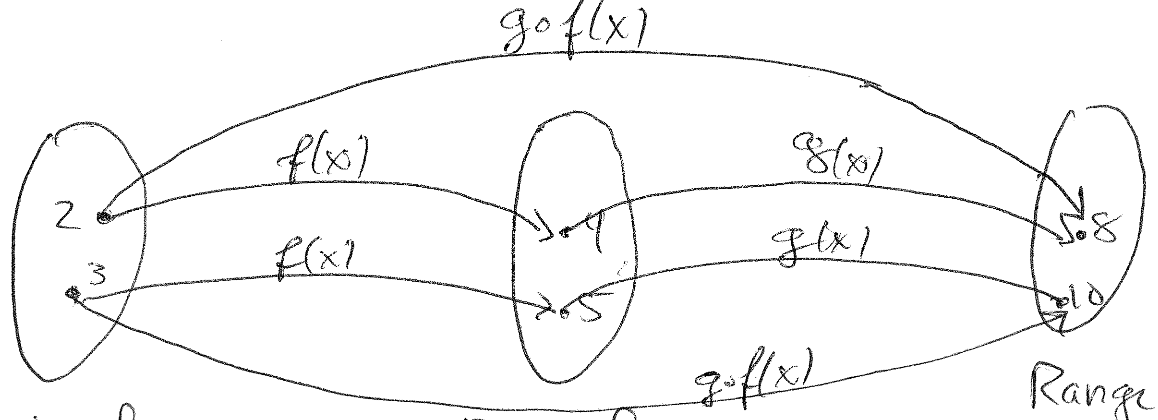
Rmk: In general $f \circ g$ is not the same as $g \circ f$.

E.g.: $f(x) = \sqrt{x}$, $g(x) = -x$

The composition $f \circ g(x)$ is only defined if we restrict the domain of $g(x)$ to $(-\infty, 0]$, because if $x > 0$, then

$f \circ g(x) = \sqrt{-x}$ is not a real number!

$f(x) = x + 2$, $g(x) = 2x$



Domain of f (all reals)

(also the domain of $g \circ f(x)$)

Range of f (all real numbers)

This is the <u>same</u> as the domain of g

Range of g (also the range of $g \circ f$)

$g \circ f(x) = 2x + 4$

<u>E.g.:</u> $f(x) = x^2$, $g(x) = x + 1$

a) Compute $g \circ f(x)$

$$g(f(x)) = g(x^2) = x^2 + 1$$

b) Compute $f \circ g(x)$

$$f(g(x)) = f(\underline{x+1}) = (x+1)^2 = x^2 + 2x + 1$$

Inverses

Eg: Local pizza shop offers \$12 cheese pizzas and \$2 for each additional topping.

$P(x) = 12 + 2x$ – this function models the price of a pizza with x toppings.

Say Bob bought a y-dollar pizza; how many toppings did he get?

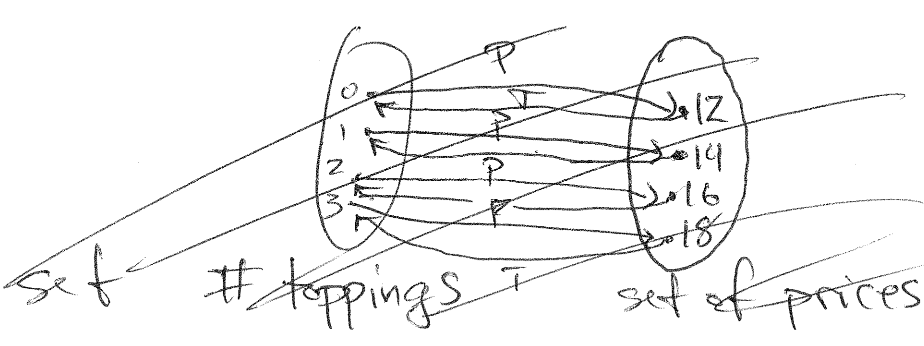
Set $y = 12 + 2x$ and solve for x.

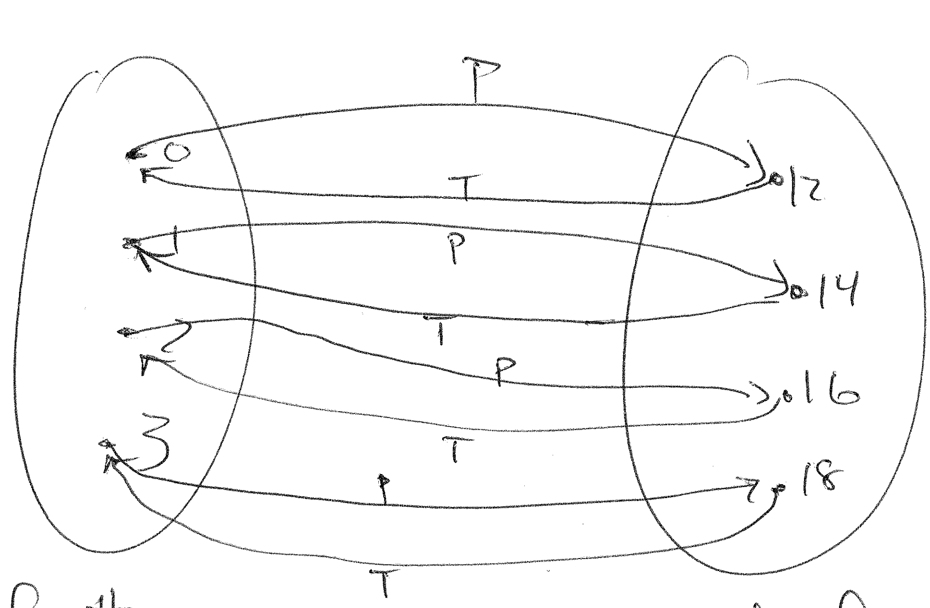
$$\Rightarrow y - 12 = 2x$$

$$\Rightarrow \frac{y-12}{2} = x$$

The function $T(y) = \frac{y-12}{2}$ (toppings function) accepts a price and outputs a number of toppings.

Visually





Set of # of toppings

Set of prices.

Two possible compositions: Apply P to a # of toppings, then apply T to the price,

OR

Apply T to a price, then apply P to the number of toppings.

$P(x) = 12 + 2x, \quad T(x) = \frac{x-12}{2}$

$$\begin{aligned} T \circ P(x) &= T(P(x)) \\ &= T(12+2x) \\ &= \frac{(12+2x)-12}{2} \\ &= \frac{2x}{2} \\ &= x. \end{aligned} \qquad \begin{aligned} P \circ T(x) &= P(T(x)) \\ &= P\left(\frac{x-12}{2}\right) \\ &= 12 + 2\left(\frac{x-12}{2}\right) \\ &= 12 + x - 12 \\ &= x. \end{aligned}$$

Def^n: If a function f has domain A and range B then its inverse function (if it exists) is the function f^{-1} (read "f inverse") with domain B and range A defined by

$$f^{-1}(y) = x \text{ if and only if } f(x) = y.$$

Eg.: $f(x) = x+2$ $\qquad f^{-1}(y) = y-2$

$f(2) = 2+2 = 4 = y \qquad f^{-1}(4) = 4-2 = 2$

$y = 2$

$f^{-1}(2) = 2-2 = 0$

$f(0) = 0+2 = 2$

Recall the definition of the logarithm

If a is a positive number,

$$\log_a(x) = y \text{ if and only if } a^y = x$$

Take $f(x) = \log_a(x)$, $f^{-1}(y) = a^y$. This statement translates to

$$f(x) = y \text{ if and only if } f^{-1}(y) = x.$$

Rmk: The definition of inverse functions is equivalent to saying

$$f\circ f^{-1}(y) = y \quad \underline{\text{and}} \quad f^{-1}\circ f(x) = x.$$

In particular,

$$\log_a(a^x) = x \quad \text{and} \quad a^{\log_a(x)} = x.$$