Solving Power Equations

The power equation $x^n = a$ has the solution

$x = \sqrt[n]{a}$ if n is odd

$x = \pm\sqrt[n]{a}$ if n is even.

If also n is even, $a<0$, then the equation has no real solutions.

$\sqrt[n]{a} = a^{1/n}$

$(\sqrt[n]{a})^n = (a^{1/n})^n = a^{1/n \cdot n} = a^1 = a$

If n is even, then we can write $n = 2m$ for some integer m. E.g. $n=10$, $n = 2(5)$. If we have an nth root of a, $\sqrt[n]{a}$, then

$$
\begin{aligned}
(-\sqrt[n]{a})^n &= (-1)^n(\sqrt[n]{a})^n \\
&= (-1)^{2m} a \\
&= ((-1)^2)^m a \\
&= 1^m a \\
&= a.
\end{aligned}
$$

If n is odd, $n-1$ is even. So then for some integer m, $n-1 = 2m$. Then $n = (n-1)+1 = 2m+1$.

If $\sqrt[n]{a}$ is an nth root of a, then

$$(-\sqrt[n]{a})^n = (-1)^n(\sqrt[n]{a})^n = (-1)^{2m+1}a = (-1)^{2m}(-1)^1 a = -a.$$

So $-\sqrt[n]{a}$ is <u>not</u> an nth root of a.

<u>E.g.:</u> $\sqrt{16} = 4$

So 4 is a solution to

$$x^2 - 16 = 0 \quad \text{(equivalent to solving } x^2 = 16\text{)}$$

because $4^2 - 16 = 16 - 16 = 0$. But also, -4 is a solution because

$$\begin{aligned}(-4)^2 - 16 &= (-1)^2 4^2 - 16 \\ &= 1 \cdot 16 - 16 \\ &= 0.\end{aligned}$$

<u>odd</u> $n = 3$

$$x^3 = 8$$

So $2 \cdot 2 = 4$, $4 \cdot 2 = 8$, gives $2^3 = 8$ and thus 2 is a solution!

$$2^3 = 8.$$

But -2 is <u>not</u> a solution because

$$(-2)^3 = (-1)^3 2^3 = (-1)^{2(1)+1} 8 = (-1)^2(-1)8 = -8 \neq 8.$$

Solving For One Variable In terms of Others

Eg.: Solve the equation

$$tx = 2t + 3x$$

for x.

Subtract $3x$ from both sides to get

$$tx - 3x = 2t$$

Factor an x out on the left to get

$$x(t-3) = 2t$$

Divide both sides by $t-3$ to get

$$x = \frac{2t}{t-3}, \quad t \neq 3.$$

Eg.: Solve $F = \frac{GmM}{r^2}$ for M.

Multiply both sides by r^2/Gm to get

$$\frac{Fr^2}{Gm} = M \quad \text{when } G \neq 0,\ m \neq 0.$$

6.2 Solving Quadratic Equations

Def^{n}: A quadratic equation in a variable, say x, has the form

$$ax^2 + bx + c = 0, \quad a \neq 0.$$

Zero-Product Property

If A and B are two real numbers, then whenever $AB = 0$, one of $A = 0$ or $B = 0$ (or both!) must hold.

Eg.: Solve the equation $x^2 = 4x + 21$.

Subtract $4x+21$ from both sides:

$$x^2 - 4x - 21 = 0$$

$$x^2 - 4x - 21 = (x-7)(x+3) = 0.$$

By the zero-product property, we know either

$$x - 7 = 0 \quad \text{or} \quad x + 3 = 0$$

So $x = 7$ or $x = -3$.

Check: $7^2 - 4(7) - 21 = 49 - 28 - 21 = 49 - 49 = 0.$

$(-3)^2 - 4(-3) - 21 = 9 + 12 - 21 = 21 - 21 = 0.$

Eg. Solve $x^2+6x+9=0$

$$x^2+2(3)x+3^2=(x+3)^2=0$$

So the only solution is $x=-3$.

Eg.: $x^2+9=0$.

By subtracting 9 from both sides we get the power equation $x^2=-9$. Since there are no real numbers that square to a negative number, this equation has <u>no</u> <u>real</u> solutions.

<u>Completing the Square</u>

Know that $(x+a)^2 = x^2+2ax+a^2$ and
$(x-a)^2 = x^2-2ax+a^2$.

Eg. $x^2+3x=0$ $\qquad x^2+3x = x(x+3) = 0$

$$x^2+2\left(\tfrac{3}{2}\right)x+\left(\tfrac{3}{2}\right)^2=\left(\tfrac{3}{2}\right)^2$$

So

$$x^2+2\left(\tfrac{3}{2}\right)x+\left(\tfrac{3}{2}\right)^2=\left(x+\tfrac{3}{2}\right)^2=\left(\tfrac{3}{2}\right)^2=\tfrac{9}{4}$$

take the square root of both sides

$$x+3 = \pm\sqrt{\tfrac{9}{4}} = \pm\frac{\sqrt{9}}{\sqrt{4}} = \pm\frac{3}{2}$$

subtract 3 from both sides

$$x = \pm\tfrac{3}{2}-3 = \pm\tfrac{3}{2}-\tfrac{6}{2} =$$