Adding & Subtracting Rational Expressions

If A, B, C are polynomials, $C \neq 0$, then

$$\frac{A}{C} + \frac{B}{C} = \frac{A+B}{C}.$$

Eg: (a)

$$\frac{3}{x-1} + \frac{x}{x+2} = \frac{x+2}{x+2}\left(\frac{3}{x-1}\right) + \frac{x-1}{x-1}\left(\frac{x}{x+2}\right)$$

$$= \frac{3(x+2) + x(x-1)}{(x-1)(x+2)}$$

$$= \frac{3x+6+x^2-x}{(x-1)(x+2)}$$

$$= \boxed{\frac{x^2+2x+6}{(x-1)(x+2)}}$$

(b)

$$\frac{1}{x^2-1} - \frac{2}{(x+1)^2} = \frac{1}{(x+1)(x-1)} - \frac{2}{(x+1)(x+1)}$$

$$= \frac{(x+1)\cdot 1}{(x+1)^2(x-1)} - \frac{2(x-1)}{(x+1)^2(x-1)}$$

$$= \frac{x+1-2x+2}{(x+1)^2(x-1)} = \frac{-x+3}{(x+1)^2(x-1)}$$

Rationalizing the Denominator

Let A, B, C be real numbers.

$$(A+B\sqrt{C})(A-B\sqrt{C}) = A^2 - \cancel{AB\sqrt{C}} + \cancel{AB\sqrt{C}} - B^2C$$
$$= A^2 - B^2C.$$

$A - B\sqrt{C}$ is called the conjugate of $A+B\sqrt{C}$ and $A+B\sqrt{C}$ is the conjugate of $A-B\sqrt{C}$.

Eg: Rationalize the denominator of $\frac{1}{1+\sqrt{2}}$

$$\frac{1}{1+\sqrt{2}} = \frac{1}{1+\sqrt{2}}\frac{(1-\sqrt{2})}{(1-\sqrt{2})} = \frac{1-\sqrt{2}}{1-2} = \frac{1-\sqrt{2}}{-1} = \sqrt{2}-1.$$

Long Division

Intuition:

$$\begin{array}{r} 4 \\ 2\overline{)9} \\ -8 \\ \hline 1 \end{array}$$

$$9 = 2(4)+1$$
$$\Rightarrow \frac{9}{2} = \frac{2(4)+1}{2}$$
$$= \frac{\cancel{2}(4)}{\cancel{2}} + \frac{1}{2}$$
$$= 4 + \frac{1}{2}.$$

E.g.: $\dfrac{6x^2-26x+12}{x-4}$

$$
\begin{array}{rl}
 & \underline{\;6x-2\qquad\qquad} \\
x-4 & \big)\, 6x^2-26x+12 \\
 & \underline{-6x^2+24x\qquad} \\
 & 0\quad -2\cancel{x}+12 \\
 & \underline{\qquad +2\cancel{x}-8} \\
 & \qquad\quad 4
\end{array}
$$

$$6x^2-26x+12-6x(x-4)$$
$$\cancel{6x^2}-26x+12-\cancel{6x^2}+24x$$
$$-2x+12-(-2)(x-4)$$
$$-2x+12-(-2x+8)$$
$$-2x+12+2x-8$$

$$6x^2-26x+12=(x-4)(6x-2)+4$$

Check:

$$(x-4)(6x-2)+4=\left[6x^2-2x-24x+8\right]+4$$
$$=6x^2-26x+12 \checkmark$$

$$\frac{6x^2-26x+12}{x-4}=\frac{(x-4)(6x-2)+4}{x-4}$$

$$\frac{6x^2-26x+12}{x-4}=6x-2+\boxed{\frac{4}{x-4}}$$

Algebra Toolkit C

Working with Equations

Def^n: An equation is a statement that two arithmetic expressions are the same.

E.g.: $4x+7=19$

The set of values for x is called the set of solutions/roots.

Operations on Equations

1. Add or subtract values from both sides.

E.g.: $$\left.\begin{aligned} 4x+7+3 &= 19+3 \\ 4x+10 &= 22. \end{aligned}\right\}$$ the "same" solution set.

2. Multiply/divide both sides by any non-zero quantity

Eg: $$4x+7=19$$
$$2(4x+7)=2\cdot 19$$
$$8x+14=38$$

Eg.: Solve ~~5(3x+~~

$(5)(3+x) = 9x$; distribute the 5

$15 + 5x = 9x$; subtract $5x$ from both sides

$15 = 4x$; divide both sides by 4

$\frac{15}{4} = x.$

Defn: A linear equation in one variable is an equation equivalent to one of the form

$$ax + b = 0.$$

Eg.: $7x - 4 = 3x + 8$; subtract $3x+8$ from both sides

$7x - 4 - (3x+8) = 0$; simplify

$7x - 3x - 4 - 8 = 4x - 12 = 0.$

$\underbrace{4}_{a}x + (\underbrace{-12}_{b}) = 0.$

$ax + b = 0$, $a \neq 0$

$ax = -b$

$x = -b/a.$

Solving ~~Rational~~ Equations w/ Rational Expressions

Eg: $2 + \frac{5}{x-4} = \frac{x+1}{x-4}$

"False Solution"

Multiply both sides by $x-4$, get

$$(x-4)\left(2 + \frac{5}{x-4}\right) = \frac{x+1}{x-4}\cdot(x-4)$$

$$2(x-4) + \frac{\cancel{(x-4)}\cdot 5}{\cancel{x-4}} = x+1 \cdot \frac{\cancel{x-4}}{\cancel{x-4}}$$ simplify

$2x - 8 + 5 = x + 1$ subtract x from both sides

$2x - 8 + 5 - x = 1$

$x - 3 = 1$ add 3 to both sides

$x = 4.$

Wrong.

There are no solutions here because

$2 + \frac{5}{x-4}$ and $\frac{x+1}{x-4}$ are undefined at $x=4$ (division by 0 is bad)