Completing the Square

$f(x) = ax^2+bx+c \leftarrow$ general form

$$\boxed{\begin{aligned} x^2+2Ax+A^2 &= (x+A)^2 \\ x^2-2Ax+A^2 &= (x-A)^2 \end{aligned}}$$

Idea: Want to put $f(x)$ in vertex form; these forms are equivalent

1. Factor a out of ~~both terms~~ the first 2 terms

$$f(x) = a\left(x^2+\frac{b}{a}x\right)+c \qquad \left(\frac{b}{a} = 2\left(\frac{b}{2a}\right)\right)$$

2. Add $\left(\frac{b}{2a}\right)^2$ and subtract $\left(\frac{b}{2a}\right)^2$ inside the parentheses

$$f(x) = a\left(\underbrace{x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2}-\left(\frac{b}{2a}\right)^2\right)+c$$

3. Factor

$$f(x) = a\left(\underbrace{x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2}-\left(\frac{b}{2a}\right)^2\right)+c$$

$$= a\left(\left(x+\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2\right)+c$$

4. Distribute the a

$$f(x) = a\left(x+\frac{b}{2a}\right)^2 \underbrace{-a\left(\frac{b}{2a}\right)^2+c}_{\text{y-coord of the vertex.}}$$

$\frac{-b}{2a}$ x-coord of vertex

Eg:

$$f(x) = x^2 + 2x + 3$$
$$= x^2 + 2(1)x + 3$$
$$= \underbrace{x^2 + 2x + 1^2} - 1^2 + 3$$
$$= (x+1)^2 - 1 + 3$$
$$= (x+1)^2 + 2.$$

vertex: $(-1, 2)$

②

$$x^2 + 2Ax + A^2 = (x+A)^2$$
$$x^2 - 2Ax + A^2 = (x-A)^2$$

Eg:

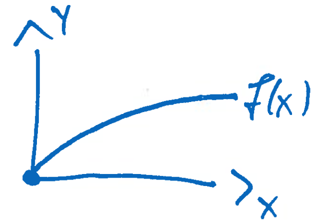
$$f(x) = 2x^2 - 3x + 5$$
$$= 2\left(x^2 - \tfrac{3}{2}x\right) + 5 = 2\left(x^2 - 2\left(\overset{A}{\tfrac{3}{4}}\right)x\right) + 5$$
$$= 2\left(\underbrace{x^2 - \tfrac{3}{2}x + \left(\tfrac{3}{4}\right)^2} - \left(\tfrac{3}{4}\right)^2\right) + 5$$
$$= 2\left(\left(x - \tfrac{3}{4}\right)^2 - \left(\tfrac{3}{4}\right)^2\right) + 5$$
$$= 2\left(x - \tfrac{3}{4}\right)^2 - 2\left(\tfrac{9}{16}\right) + 5$$
$$= 2\left(x - \tfrac{3}{4}\right)^2 - \tfrac{9}{8} + \tfrac{40}{8}$$
$$= 2\left(x - \tfrac{3}{4}\right)^2 + \tfrac{31}{8}.$$

vertex $\left(\tfrac{3}{4}, \tfrac{31}{8}\right).$

$$x^2 + bx = \underbrace{x^2 + bx + \left(\tfrac{b}{2}\right)^2} - \left(\tfrac{b}{2}\right)^2$$
$$= \left(x + \tfrac{b}{2}\right)^2 - \left(\tfrac{b}{2}\right)^2$$

Domain of $\sqrt{x} = f(x)$ is

$$\{x \in \mathbb{R} \mid 0 \le x\}$$



The question

$f \circ g(x) = f(g(x))$

"What is the domain of $\sqrt{g(x)}$?"

is equivalent to asking the question

"What are the values in the domain of g such that $0 \le g(x)$?"

E.g: What is the domain of

$$\sqrt{5x+7} \; ?$$

$5x+7$ has domain $\mathbb{R}$: we need to solve the linear inequality

$$0 \le 5x+7$$

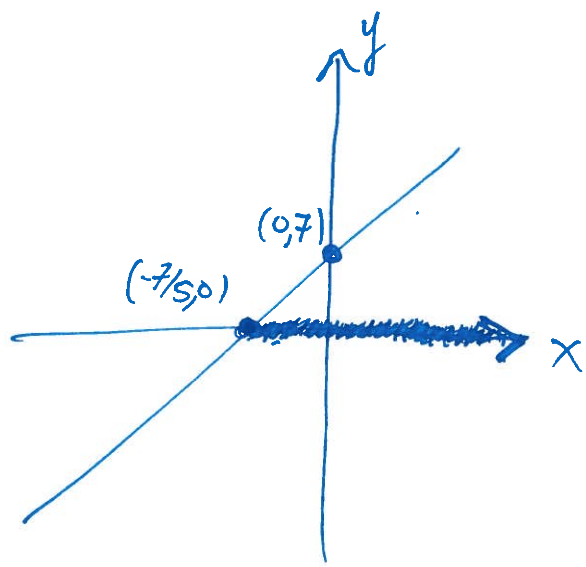
Subtract 7 from both sides

$$-7 \le 5x$$

Divide both ~~is~~ sides by 5

$$\frac{-7}{5} \le x.$$

$$[-7/5, \infty)$$



E.g: What is the domain of

$$\sqrt{9-x^2}\ ?$$

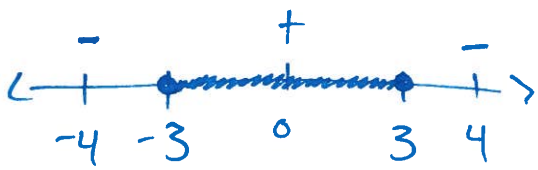
Domain $9-x^2$: $\mathbb{R}$

Domain $\sqrt{9-x^2}$: x such that $0 \le 9-x^2$

To solve the non-linear inequality, first solve

$$0 = 9-x^2 = (3-x)(3+x)$$

So either $3 = x$ or $x = -3$.



$[-3,3]$ - domain of $\sqrt{9-x^2}$

$9-(-4)^2 = 9-16 < 0$

$9-4^2 = 9-16 < 0$

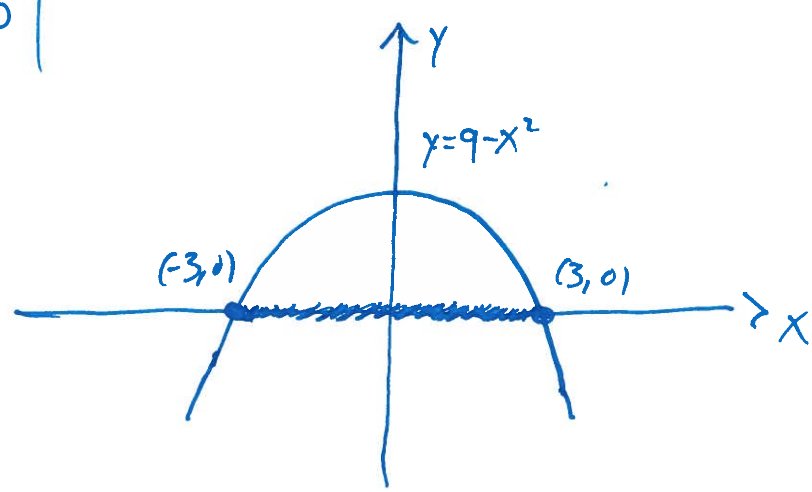
$9-0^2 = 9-0 = 9 > 0$

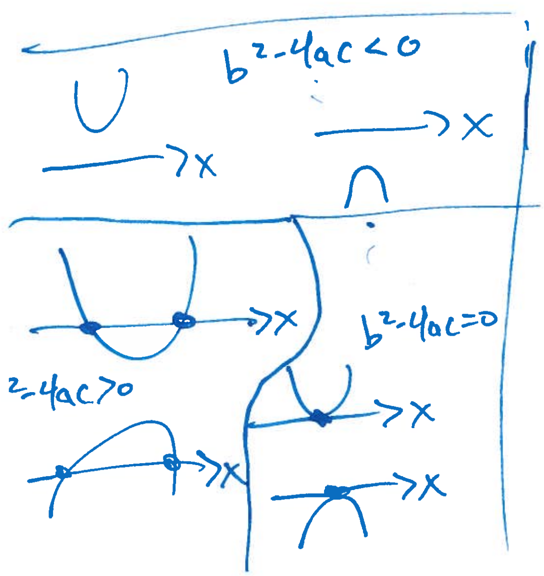
Geometrically $9-x^2 = -x^2+0x+9$

$= (x+0)^2+9$

Vertex: $(0,9)$

x-intercepts: $(-3,0)$, $(3,0)$

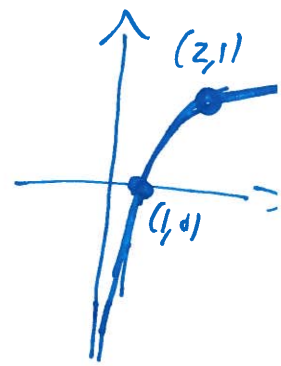




E.g: What is the domain of

$$\sqrt{\log_2(x+7)}\ ?$$

The logarithm satisfies:



$$\log_2(1)=0$$

$1<x$, ~~that~~ then $\log_2(x)>0$

$0<x<1$, then $\log_2(x)<0$

undefined for $x<0$

Need $\log_2(x+7)\geq 0$, so need $1\leq x+7$

$\Rightarrow -6\leq x$

$$[-6,\infty)$$

Geometrically: