4.1:

Jargon: when $a=10$, the book will often write
$$\log(x) := \log_{10}(x).$$

Basic Properties of Logarithm

Recall: $\log_a(x) = y$ is equivalent to saying $x = a^y$.

1. $\log_a(1) = \log_a(a^0) = 0$
2. $\log_a(AB) = \log_a(A) + \log_a(B)$

Pf: ~~$y = \log_a(AB) \Leftrightarrow a^y = AB$ (Recall)~~

$a^{\log_a(AB)} = AB$

$a^{\log_a(A) + \log_a(B)} = a^{\log_a(A)} a^{\log_a(B)} = AB$ } these are the same $\Rightarrow \log_a(AB) = \log_a(A) + \log_a(B)$

3. $\log_a\left(\frac{A}{B}\right) = \log_a(A) - \log_a(B)$

Pf: $a^{\log_a(A/B)} = \frac{A}{B}$

$a^{\log_a(A) - \log_a(B)} = \frac{a^{\log_a(A)}}{a^{\log_a(B)}} = \frac{A}{B}$ } since these are the same, this says $\log_a\left(\frac{A}{B}\right) = \log_a(A) - \log_a(B)$.

4. $\log_a(A^c) = c\log_a(A)$

Pf: $\log_a(A^c) = \log_a(A \cdot A^{c-1}) \overset{(2)}{=} \log_a(A) + \log_a(A^{c-1})$

$= \log_a(A) + \log_a(A A^{c-2})$

$\overset{(2)}{=} \log_a(A) + \log_a(A) + \log_a(A^{c-2})$

$\vdots$

$$= \underbrace{\log_a(A) + \log_a(A) + \ldots + \log_a(A)}_{C \text{ terms}}$$

$$= C\log_a(A).$$

5 (Change of Base): $0 < a, b \neq 1$.

$$\log_b(A) = \frac{\log_a(A)}{\log_a(b)} \leftarrow \text{useful for computing things.}$$

<u>E.g.</u>: Evaluate the expressions

a) $\log_4(2) + \log_4(32)$

b) $\log_2(80) - \log_2(5)$

c) $-\frac{1}{3}\log_2(8)$

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a) $\log_4(2) + \log_4(32) = \log_4(2 \cdot 32) = \log_4(64) = \log_4(2^6)$

$$= \log_4((2^2)^3) = \log_4(4^3) = 3.$$

b) $\log_2(80) - \log_2(5) = \log_2\left(\frac{80}{5}\right) = \log_2\left(\frac{8 \cdot 10}{5}\right) = \log_2(8 \cdot 2)$

$$= \log_2(16) = \log_2(2^4) = 4.$$

c) $-\frac{1}{3}\log_2(8) = \log_2(8^{-1/3}) = \log_2\left(\frac{1}{8^{1/3}}\right) = \log_2\left(\frac{1}{\sqrt[3]{8}}\right) = \log_2\left(\frac{1}{2}\right)$

$$= \log_2(1) - \log_2(2) = 0 - 1 = -1.$$

(or $\log_2(2^{-1}) = -1$)

E.g.: Expand

a) $\log_2(6x)$ b) $\log_4\left(\frac{z^2}{y}\right)$ c) $\log_5(x^3y^6)$ d) $\log\left(\frac{ab}{\sqrt[3]{c}}\right)$

=

a) $\log_2(6)+\log_2(x)$

b) $\log_4\left(\frac{z^2}{y}\right) = \log_4(z^2) - \log_4(y)$

$= 2\log_4(z) - \log_4(y)$

c) $\log_5(x^3y^6) = \log_5(x^3) + \log_5(y^6)$

$= 3\log_5(x) + 6\log_5(y)$

d) $\log(ab) - \log(\sqrt[3]{c}) =$

$= \log(a) + \log(b) - \log(c^{1/3})$

$= \log(a) + \log(b) - \frac{1}{3}\log(c)$;

E.g: Combine.

a) $3\log(x) + 2\log(x-5)$

$\shortparallel$

$\log(x^3) + \log((x-5)^2)$

$\shortparallel$

$\log(x^3(x-5)^2)$

b) $3\log(s) - \frac{1}{2}\log(t+1)$

$\shortparallel$

$\log(s^3) - \log(\sqrt{t+1})$ $(t+1)^{1/2} = \sqrt{t+1}$

$\shortparallel$

$\log\left(s^3/\sqrt{t+1}\right)$

E.g: Compute $\log_8(5)$

$$\log_8(5) = \frac{\log(5)}{\log(8)} \approx 0.77398\ldots$$

4.4: The natural exponential and logarithmic functions.

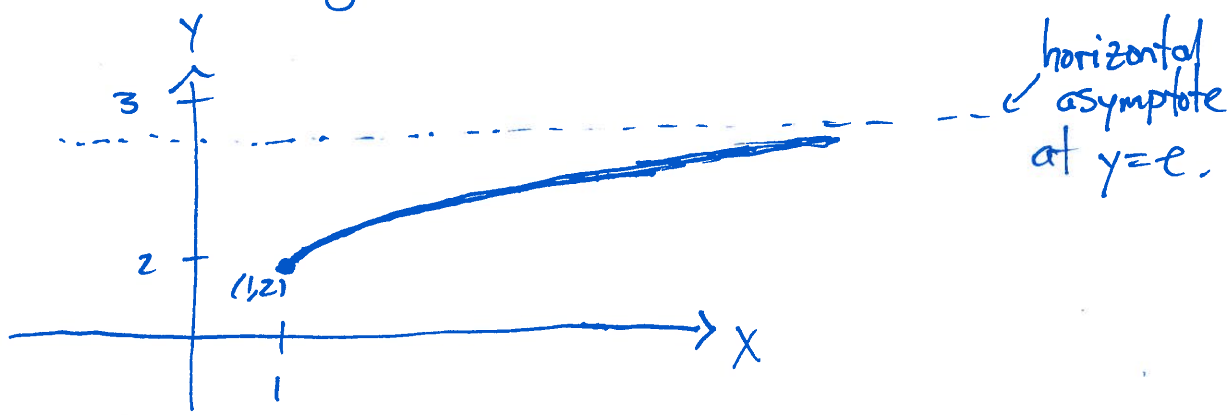
The number e: the number e is (properly) defined as

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n =: e$$

The graph of

$$f(x) = \left(1+\frac{1}{x}\right)^x$$

should look something like



One natural application is to interest. (compounding)

Recall: $A(t) = P\left(1+\frac{r}{n}\right)^{nt}$; P - principal, n - # compounding periods

r - interest rate.

What happens if # of compounding periods is "infinite"

This is called "continuously compounding interest"

Modeled by $\boxed{A(t) = Pe^{rt}}$

Exponential Growth/Decay:

The exponential function

$$f(t) = Ce^{rt}$$

models exponential growth if $r>0$ or ~~and~~ decay if $r<0$.

The value r is the instantaneous growth rate (or decay, $r<0$) expressed as a proportion of the population per time unit.

$$\frac{f(t+1)-f(t)}{f(t)} = \frac{Ce^{r(t+1)} - Ce^{rt}}{Ce^{rt}}$$

$$= \frac{Ce^{rt+r} - Ce^{rt}}{Ce^{rt}}$$

$$= \frac{Ce^{rt}e^{r} - Ce^{rt}}{Ce^{rt}}$$

$$= \frac{\cancel{Ce^{rt}}(e^{r}-1)}{\cancel{Ce^{rt}}} = e^{r}-1. = a-1$$

growth rate

$a = e^{r}$, $f(t) = Ce^{rt} = C(e^{r})^{t} = Ca^{t})$

old type growth factor

Logarithm (Natural): $\ln(x) := \log_e(x)$.

Take an exponential function

$$f(x) = Ca^{x}$$

Let $r = \ln(a)$, $e^{r} = e^{\ln(a)} = e^{\log_e(a)} = a$

$$f(x) = Ca^{x} = C\left(e^{\ln(a)}\right)^{t} = Ce^{\ln(a)t}.$$

The instantaneous growth/decay rate for $f(x) = Ca^{x}$ is just

$$r = \ln(a).$$