4.6 Working with Functions: Composition and Inverse

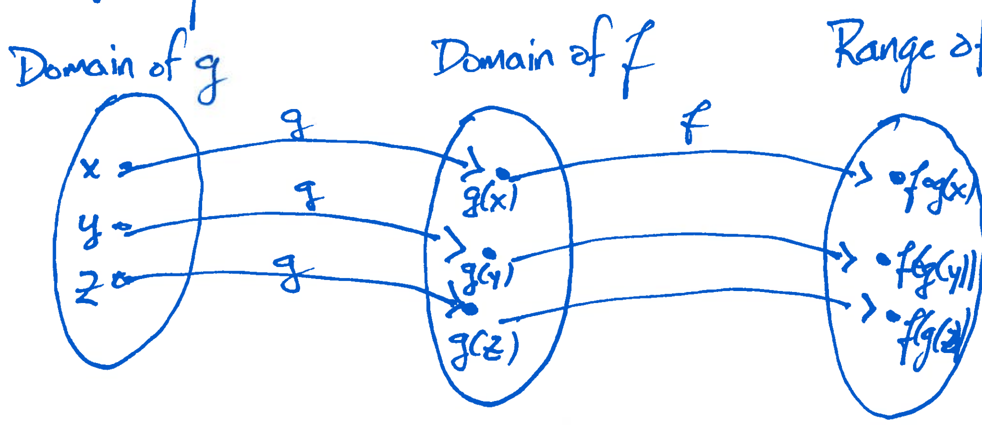
The composition of two functions, f and g, called

"the composition of f with g"

is

$$f \circ g(x) = f(g(x))$$

provided that the range of g is contained in the domain of f.



E.g. $f(x) = x + 2, \quad g(x) = 3x$

Think about $f \circ g(x)$ as the function "multiply by 3, then add 2"

$$f(g(5)) = f(3 \cdot 5) = f(15) = 15 + 2 = 17.$$

You could also think of this as the function

$$f(g(x)) = f(3x) = 3x + 2$$

E.g.:

$$g(f(x)) = g(x+2) = 3(x+2) = 3x+6.$$

which is the same as the function
"add 2, then multiply by 3".

order matters.

Warning: Function composition is non-commutative in general; that is to say (generally)

$$f\circ g \neq g\circ f$$

E.g.: $f(x) = x^2$, $g(x) = x+1$

$$f\circ g(x) = f(g(x)) = f(x+1) = (x+1)^2 = x^2+2x+1.$$

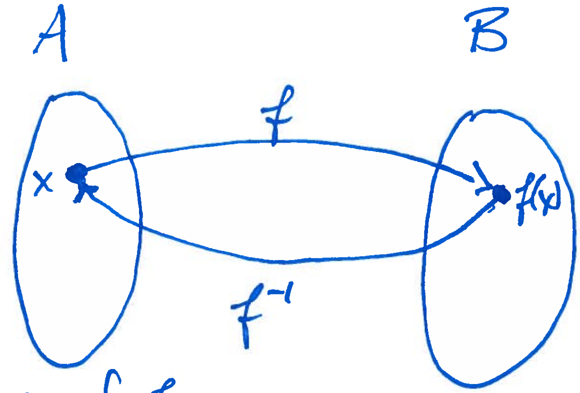
$$g\circ f(x) = g(f(x)) = g(x^2) = x^2+1$$

Here also $g\circ f(x) \neq f\circ g(x)$.

The composition inverse of a function:

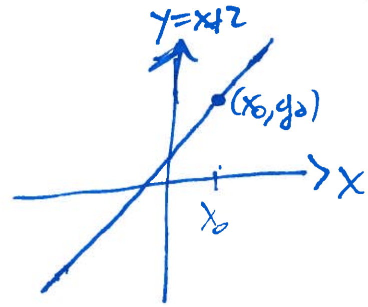
If a function f has domain A and range B, then its inverse function (if it exists) is the function f^{-1} ($\neq \frac{1}{f}$) with domain B and range A such that

$$f\circ f^{-1}(x) = x \text{ and } f^{-1}\circ f(x) = x,$$



Domain of f | Range of f
Range of f^{-1} | Domain of f^{-1}

<u>E.g.</u>: $f(x) = x+2$; Does f have an inverse, and if so, what is it?



$$\text{Domain}(f) = \mathbb{R}, \quad \text{Range}(f) = \mathbb{R}$$

$$f^{-1}(x) = x-2$$

<u>Check</u>: $f^{-1}\circ f(x) = f^{-1}(f(x)) = f^{-1}(x+2) = (x+2)-2 = x.$

$$f\circ f^{-1}(x) = f(f^{-1}(x)) = f(x-2) = (x-2)+2 = x.$$

<u>E.g.</u>: $f(x) = \pi x$, $f^{-1}(x) = \frac{x}{\pi}$

$$f^{-1}(f(x)) = f^{-1}(\pi x) = \frac{\pi x}{\pi} = x$$

$$f(f^{-1}(x)) = f\left(\frac{x}{\pi}\right) = \pi\left(\frac{x}{\pi}\right) = x.$$

E.g.: $f(x) = 3x+7$; what is its inverse?

Start out by setting

$$y = 3x+7$$

solve for x in terms of y. Subtract 7 from both sides:

$$y-7 = 3x$$

Divide both sides by 3:

$$\frac{y-7}{3} = x$$

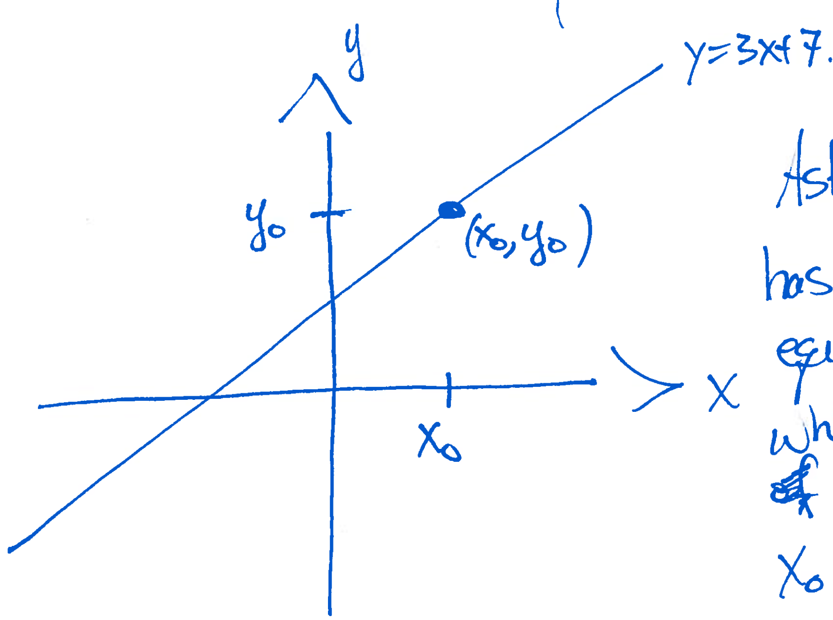
So

$$f^{-1}(x) = \frac{x-7}{3} = \frac{1}{3}x - \frac{7}{3}.$$

Check:

$$f \circ f^{-1}(x) = f\left(\frac{x-7}{3}\right) = 3\left(\frac{x-7}{3}\right) + 7 = x - 7 + 7 = x.$$

$$f^{-1} \circ f(x) = f^{-1}(3x+7) = \frac{(3x+7)-7}{3} = \frac{3x}{3} = x.$$



Asking if this function has an inverse is equivalent to asking whether for any value ~~of~~ y_0, is there a value x_0 such that $y_0 = f(x_0)$?

Eg: $f(x) = x^2$: Is there a <u>function</u> $f^{-1}(x)$ such that f^{-1} is the inverse of f? I.e.

$$f^{-1} \circ f(x) = x, \quad f \circ f^{-1}(x) = x.$$

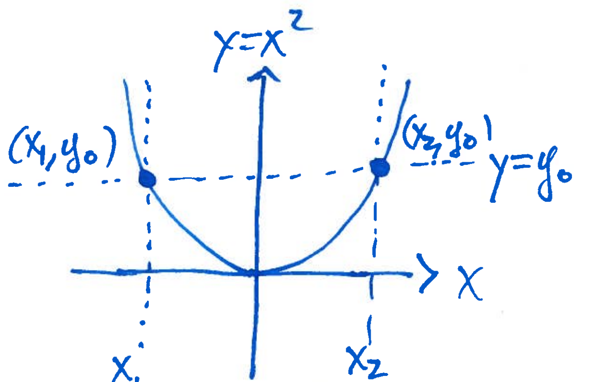
$g(x) = \sqrt{x}$:

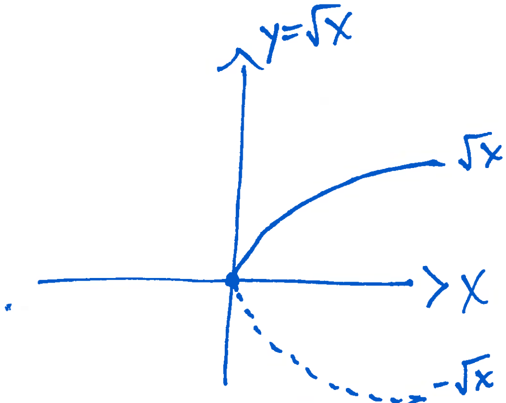
$$f \circ g(x) = f(\sqrt{x}) = (\sqrt{x})^2 = x$$
$$g \circ f(x) = g(x^2) = \sqrt{x^2} = x.$$

However, this is <u>not</u> the inverse of $f(x) = x^2$ on the whole real line... why?

There's a subtle lie here... we need

$$\text{Domain}(f) = \text{Range}(g), \quad \text{Domain}(g) = \text{Range}(f)$$



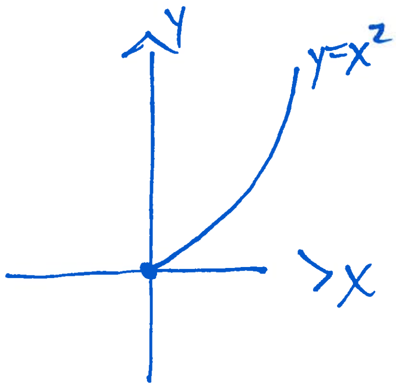


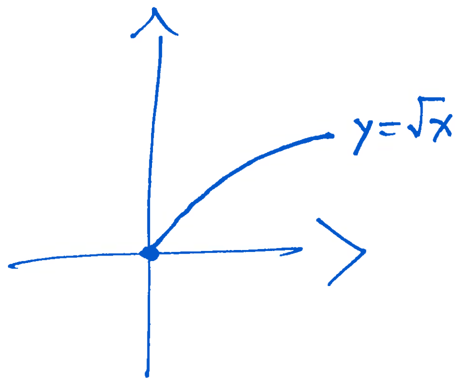
$\text{Domain}(f) = \mathbb{R}$ $\qquad$ $\text{Domain}(g) = \{0 \le x\}$

$\text{Range}(f) = \{0 \le x\}$ $\qquad$ $\text{Range}(g) = \{0 \le x\}$

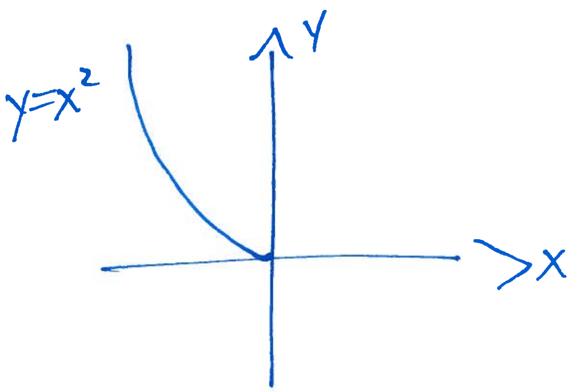
Asking the question: given some value y, can we choose a <u>unique</u> value of x such that $y = f(x) = x^2$?

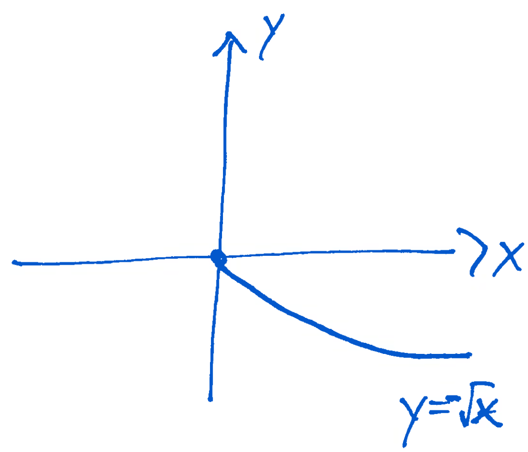
If we restrict the domain of f to $\{0 \leq x\}$, then $\sqrt{x}$ is the inverse of this function.





If we restrict the domain of f to $\{x \leq 0\}$, then $-\sqrt{x}$ is the inverse of this function





$$g \circ f(-2) = g((-2)^2) = g(4) = \sqrt{4} = 2$$

$$-\sqrt{f(-2)} = -\sqrt{4} = -2.$$

The Horizontal Line Test: If any horizontal line passes through at most one point on the graph of f, then f has an inverse. If not, then f does not have an inverse (at least on its natural domain).

This poses problems for functions involving even powers (polynomials), but not for odd powers.

Eg: $f(x) = 2x^3+1$; find its inverse

$y = 2x^3+1$ ← solve for x in terms of y; done.

$\Rightarrow y-1 = 2x^3$

$\Rightarrow \frac{y-1}{2} = x^3$

$\Rightarrow \sqrt[3]{\frac{y-1}{2}} = x.$

$f^{-1}(x) = \sqrt[3]{\frac{x-1}{2}}$

Eg: $f(x) = x^2+1$

$y = x^2+1$

$\Rightarrow y-1 = x^2$

$\Rightarrow \pm\sqrt{y-1} = x.$ ← not a function

E.g.: $f(x) = \frac{3x}{x+2}$; find the inverse

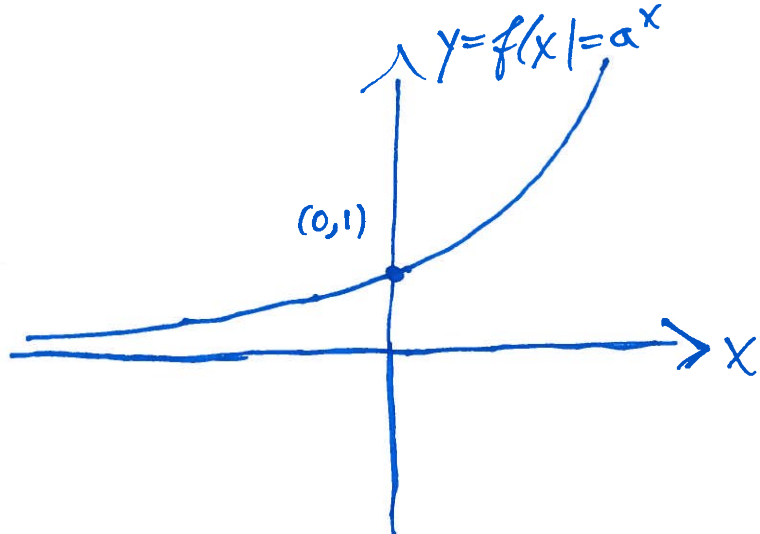
$y = \frac{3x}{x+2} \Rightarrow y(x+2) = 3x$

$\Rightarrow yx+2y = 3x$

$\Rightarrow 2y = 3x - yx = x(3-y)$

$\Rightarrow \frac{2y}{3-y} = x, \quad f^{-1}(x) = \frac{2x}{3-x}.$

If $f(x) = a^x$, what is $f^{-1}(x)$? Does it exist? (8)



This visibly passes the horizontal lines test, so it does have an inverse. We call this function (the inverse)

$$\log_a(x) = f^{-1}(x)$$

called the logarithm with base a.

In particular

$$f^{-1} \circ f(x) = f^{-1}(a^x) = \log_a(a^x) = x.$$

$$f \circ f^{-1}(x) = f(\log_a x) = a^{\log_a(x)} = x.$$

The statement

$$y = \log_a(x)$$

is equivalent to

$$a^y = x.$$

Eg.: $\log_3(27) = ?$ $\Leftrightarrow$ "to what power do we raise 3 to get 27?"

$27 = 3 \cdot 3 \cdot 3 = 3^3$, so $\log_3(27) = \log_3(3^3) = 3$.