Exponential Decay Models

Exponential decay is modeled by a function of the form

$$f(x) = Ca^x \qquad 0 < a < 1$$

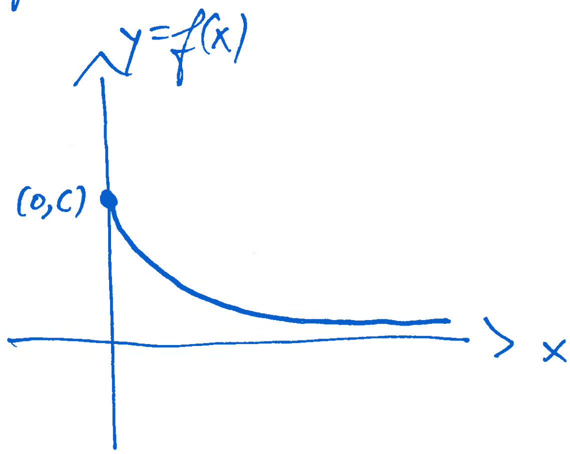
The variable x is the number of time periods

The decay factor is a.

The decay rate is $r = a - 1$ $\quad (r+1 = a)$

Rmk: r is always negative

The graph of f has the general shape



Eg.: 75mg drug. 30% expelled from the body each hour.

a) Find exp. decay model for amt of drug after x hours.

Given $r = -30\% = -0.3$, $C = 75$

$\Rightarrow a = r + 1 = -0.3 + 1 = 0.7$

$f(x) = 75(0.7)^x$

b) Predict how much remains after 4 hours.

$$f(4) = 75(0.7)^4 \approx 18.008 \text{ mg.}$$

E.g.: ½-life of Radium-226 is 1600 years. A 50g sample is stored & monitored.

a) Find a function that models the mass, $m(x)$, after x half-lives.

$$m(x) = 50\left(\tfrac{1}{2}\right)^x$$

b) Use the ~~model~~ the model to predict the amt of Radium-226 after 4000 years.

$$\frac{4000 \text{ years}}{1600 \text{ years/half-life}} = 2.5 \text{ half lives.}$$

$$m(2.5) = 50(1/2)^{2.5} \approx 8.84.$$

$$m(y) = 50\left(\tfrac{1}{2}\right)^{y/1600}$$

y = # of years elapsed.

3.2: Exponential Models: Comparing Rates

Changing the Time Period

growth factors:

Daily: a	Weekly: a^7
Weekly: b	Daily: $b^{1/7}$
Monthly: c	Yearly: c^{12}
Yearly: d	Monthly: $d^{1/12}$

Aa^x	Aa^{7x}
Bb^x	$Bb^{x/7}$
Cc^x	Cc^{12x}
Dd^x	$Dd^{x/12}$

E.g: 30 min growth rate of bacteria is 0.85
Find the 1-hour growth rate.

30 min. growth factor

$$a_{30} = 1 + r_{30} = 1 + .085 = 1.85$$

60 min/hour growth factor $\Rightarrow a_{60} = (a_{30})^2 = (1.85)^2 \approx 3.4$

So the 60 minute growth rate is

$$r_{60} = a_{60} - 1 = 3.4 - 1 = 2.4$$

E.g.: 20 chinchillas, after 3 years 128. Assume exp. growth.

a) Find the 3-year growth factor

$P(t) = P_0 a^t \leftarrow$ function of t years.

Given: $P(0) = P_0 = 20$, $P(3) = 128 = P_0 a^3$

$$\frac{P(3)}{P(0)} = \frac{128}{20} = \frac{P_0 a^3}{P_0} = a^3 \leftarrow \text{3-year growth factor}$$

b) Find the 1-year growth factor

$$\boxed{a^3 = \frac{128}{20}}$$

$$\Rightarrow \sqrt[3]{a^3} = a = \sqrt[3]{\frac{128}{20}} \approx \sqrt[3]{6.4} \approx 1.86$$

1 year growth factor

$\Rightarrow P(t) = 20(1.86)^t$, t years.

Eg.: 100 bacteria, count doubles every 5-hour time period. Find an exponential growth model for # of bacteria

(a) x time periods after infection

b) t hours after infection.

a) $P(x) = \underline{100} \cdot 2^x$

b) $a^5 = 2^1$, where a is the hourly growth factor

$$\sqrt[5]{a^5} = a = \sqrt[5]{2} \approx 1.15$$

$$P(t) = 100 \cdot (1.15)^t$$

Alternatively:

$t = 5x \Rightarrow x = \frac{1}{5}t$

Since 1 time period is 5 hours

$$P(x) = 100 \cdot 2^x = 100 \cdot 2^{\frac{1}{5}t} \approx 100(1.15)^t.$$

because

$$2^{\frac{1}{5}t} = (2^{\frac{1}{5}})^t = (\sqrt[5]{2})^t \approx (1.15)^t.$$

Eg.: Bacteria A & B

A doubles every 5 hours

B ~~doubles~~ triples every 7 hours

a) Find 1-hour growth rate of each

b) Which one grows faster?

Type A is the last example: 1-hour growth factor is 1.15. One-hour growth rate is

$$1.15 - 1 = 0.15. \quad (15\% \text{ each hour})$$

Type B

Let a be the hourly growth rate

$$\Rightarrow a^7 = 3$$

$$\Rightarrow a = 3^{1/7} \approx 1.17$$

$$\Rightarrow r = 1.17 - 1 = 0.17 \quad (17\% \text{ each hour}).$$

So type B grows faster.

Growth of an Investment: Compound Interest

If P units of money are invested at an annual interest rate of r, compounded n times each year, then the amount $A(t)$ after t years is given by

$$A(t) = P(1 + r/n)^{nt}$$

E.g: $r = 10\%$, $n = 2$, $P = 100$

$t = 0$: 100

After 6 months: $100(1 + \frac{.1}{2}) = 100(1.05) = 100 + 100(0.05)$

$= 105$

After another 6 months: $105(1.05) = 105 + 105(.05)$

$= \ldots$

E.g.: \$5000, invest in 3-year CD. Two choices:

A: 5.50% /year, compounded twice a year.

B: 5.50% /year, compounded daily

Which is better?

$r = .055$, $A(3) = ?$, $A(t) = P\left(1+\frac{r}{n}\right)^{nt}$

$P = 5000$

$A: n = 2$	$B: n = 365$
$A(t) = 5000\left(1+\frac{0.055}{2}\right)^{2t}$	$A(t) = 5000\left(1+\frac{0.055}{365}\right)^{365 \cdot t}$
$A(3) = \$5883.84$	$A(3) = \$5896.89.$

B is the better choice.