2.7 Linear Equations: Where Lines Meet

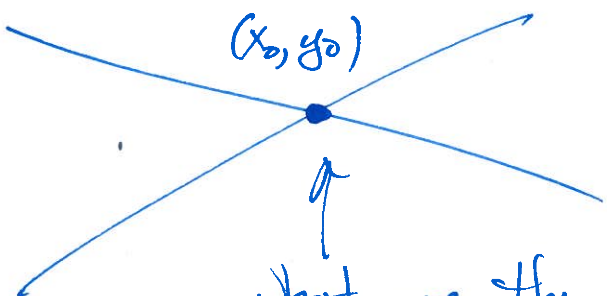
To find where the graph of the linear functions

$$y = m_1 x + b_1 \quad \text{and} \quad y = m_2 x + b_2$$

intersect, we need to solve for x in the equation

$$m_1 x + b_1 = m_2 x + b_2 .$$

Pictorially:



what are the coordinates of this point?

Get the x's on one side: subtract $m_2 x$ from both sides to get

$$m_1 x - m_2 x + b_1 = b_2$$

subtract b_1 from both sides to get

$$m_1 x - m_2 x = b_2 - b_1$$

factor out an x on the left to get

$$(m_1 - m_2) x = b_2 - b_1$$

Provided these lines are not parallel, $m_1 - m_2 \neq 0$, so we can divide both sides by $m_1 - m_2$ to get

$$x = \frac{b_2 - b_1}{m_1 - m_2} .$$

Once you have the x-coordinate, plug into either line to get the y-coordinate.

Eg.: $f(x) = 5x-8, \quad g(x) = 3x+2$

The x-coordinate of the point of intersection is

$$x = \frac{-8-2}{3-5} = \frac{-10}{-2} = 5$$

$$5x-8 = 3x+2$$

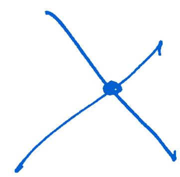
$$\Rightarrow 5x-3x-8 = 2$$

$$\Rightarrow 2x-8 = 2$$

$$\Rightarrow 2x = 2+8 = 10$$

$$\Rightarrow x = 10/2 = 5.$$

Eg.: $y = 8p-10, \quad y = -3p+15$



$$8p-10 = -3p+15$$

$$\Rightarrow 11p = 25$$

$$\Rightarrow p = \frac{25}{11}$$

3. Exponential Functions

Exponential Growth model is a function of the form

$$f(x) = Ca^x, \quad C \in \mathbb{R}, \quad 1 < a.$$

Eg: $f(x) = 2^x$ — this function effectively takes in a number and multiplies 2 by itself that many times.

$f(0) = 2^0 = 1$ $\qquad$ $f(2) = 2^2 = 2\cdot 2 = 4$

$f(1) = 2^1 = 2$ $\qquad$ etc.

$f(-1) = 2^{-1} = 1/2$

$f(-2) = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$

In particular, these kinds of exponential functions always have a "common" point

$$f(x) = a^x, \quad f(0) = a^0 = 1.$$

For a general exponential function

$$f(x) = Ca^x, \quad f(0) = Ca^0 = C$$

We call this value C the <u>initial value</u>.

The variable x will often stand for a number of <u>time periods</u>.

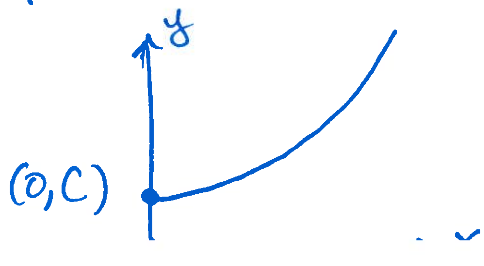
The base a is called the <u>growth factor</u>; this is the factor by which $f(x)$ is multiplied to obtain $f(x+1)$.

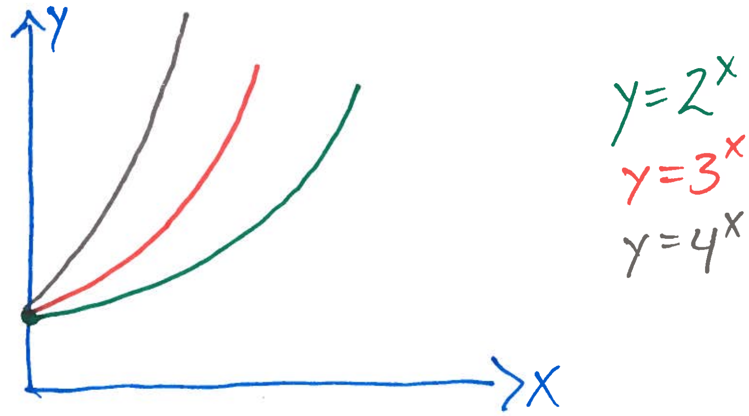
$$f(x+1) = Ca^{x+1} = \underbrace{Ca^x}_{f(x)}a = af(x)$$

When one assumes that the exponential function

$$f(x) = Ca^x$$

~~rep~~ models some phenomenon in variable time, the graph looks like





E.g.: Find a model for the following exponential growth situations

a) A bacterial infection starts w/ 100 bacteria and triples every hour.

$$P(h) = P_0 a^h$$

Given $P_0 = 100$.

After 1 hour, $P(1) = 300$
After 2 hours, $P(2) = 900$
After 3 hours, $P(3) = 2700$
⋮

$$P(h) = 100 \cdot 3^h$$

b) A pond is stocked w/ 5800 fish and each year the fish population increases by 20%.

$$P(y) = P_0 a^y, \quad P_0 = 5800.$$

$$20\% = \frac{20}{100} = \frac{1}{5}$$ 1.2

fter year: $P_0 + \frac{1}{5}P_0 = P_0(1 + \frac{1}{5}) = P_0(\frac{6}{5})$

fter years: $P_0(\frac{6}{5}) + \frac{1}{5}P_0(\frac{6}{5}) = P_0(\frac{6}{5})(1 + \frac{1}{5}) = P_0(\frac{6}{5})(\frac{6}{5})$

$$= P_0\left(\frac{6}{5}\right)^2$$

⋮

This tells us that $a = 4/5$. So

$$P(y) = 5800(4/5)^y.$$

E.g: 20 chinchillas. After 3 years there are 128 chinchillas. Assume exp. growth. Find the 3-year growth factor

Want some sort of $P(y) = P_0 a^y$. Given

$$P_0 = 20 = P(0)$$

$$P(3) = 128 = 20a^3$$

$$a^3 = \frac{20a^3}{20} = \frac{128}{20} = 6.4$$

The growth rate for an exponential function is

$$r = \frac{f(x+1) - f(x)}{f(x)}$$

$$a^{x+1} = a^x \cdot a$$

$$= \frac{Ca^{x+1} - Ca^x}{Ca^x}$$

$$= \frac{(Ca^x)a - Ca^x}{Ca^x}$$

$$= \frac{\cancel{Ca^x}(a-1)}{\cancel{Ca^x}} = a-1.$$

growth factor

Equivalently, $a = 1 + r$. This is a defining property of exponential functions

Rmk For exponential growth, the growth rate is always positive, so

$$a = 1 + r > 1.$$

E.g.: 50 rabbits. Population increases by 60% each year.

a) Find a function P of x years modelling the population.

Need: $P(x) = P_0 a^x$

Given: $P_0 = 50$, $r = 0.6$, $a = 1 + r$

$$P(x) = 50(1.6)^x$$

b) How many rabbits after 8 years?

$$P(8) = 50(1.6)^8 \approx 2147.48$$

Roughly, 2147 rabbits.