2.2 : Linear Functions: Constant rate of Change 10/10/17

Def$^{\underline{n}}$: A linear function is a function of the form

$$f(x) = mx + b$$

where b, m are ~~a~~ real numbers.

The graph of a linear function is a line.

The number~~s~~ m is called the slope of the line, and the number b is usually called the y-intercept or the vertical intercept.

The slope tells you the vertical change/net change over a unit increase in the independent variable is m.

For $a, a+1$ the net change of $f(x) = mx + b$ is

$$\begin{aligned} f(a+1) - f(a) &= m(a+1) + b - (ma + b) \\ &= \cancel{ma} + m + \cancel{b} - \cancel{ma} - \cancel{b} \\ &= m \end{aligned}$$

These are exactly the constant "first differences" in Ch. 1.

In fact, for any input values, say a_1, a_2, $a_1 < a_2$

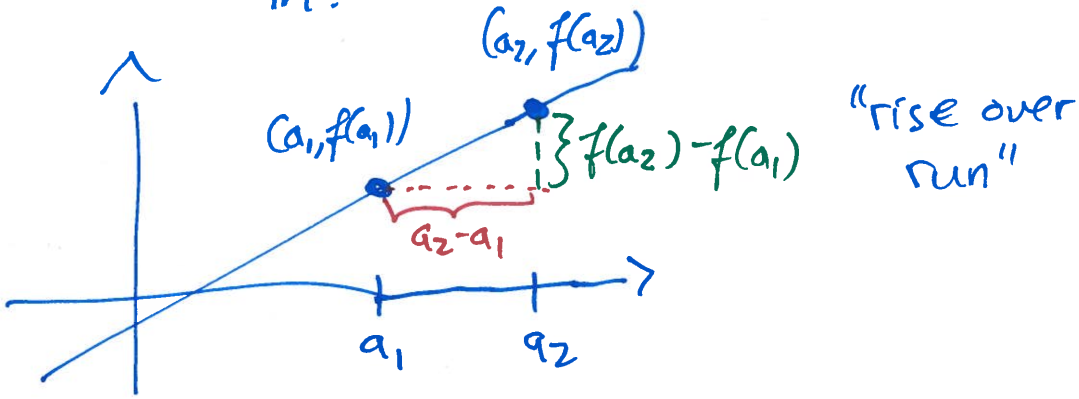
$$\frac{f(a_2) - f(a_1)}{a_2 - a_1} = \frac{(m\overset{a_2}{a_1} + b) - (m\overset{a_1}{a_2} + b)}{a_2 - a_1}$$

$$= \frac{m\overset{a_2}{a_1} + \cancel{b} - m\overset{a_1}{a_2} - \cancel{b}}{a_2 - a_1}$$

$$= \frac{m\overset{a_2}{a_1} - m\overset{a_1}{a_2}}{a_2 - a_1}$$

$$= \frac{m(\overset{a_2}{a_1} - \overset{a_1}{a_2})}{a_2 - a_1}$$

$$= m.$$

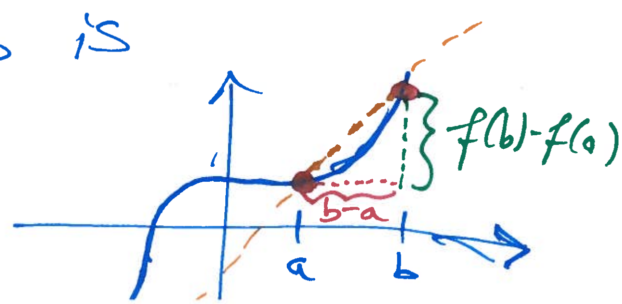


This is the average rate of change of the function $f(x)$. For lines, this is <u>always</u> constant and it is in fact the defining property of a line.

2.1: The average rate of change of an arbitrary function, f, from a to b is

$$\frac{f(b) - f(a)}{b - a}$$

Graphically, this is



The average rate of change of f from a to b is the slope of the orange line above, which is called the <u>secant line</u>.

E.g.: A car drives at a constant speed for 2 hours, and travels 60 miles; what is the speed of the car at time $t = 1.5$ hours?

Say at time 0 the car is 0 miles away from its starting location: these give 2 points, $(0,0)$ and $(2,60)$

The slope of the line is the speed at any point!

$$\frac{60-0}{2-0} = \frac{60}{2} = 30 \text{ mph.}$$

The line modeling the position of this car at time t is (with respect to the starting point

$$s(t) = \frac{60}{2}t + 0 = 30t.$$

For any time, t, the average rate of change <u>is</u> the speed of the car: 30 mph.

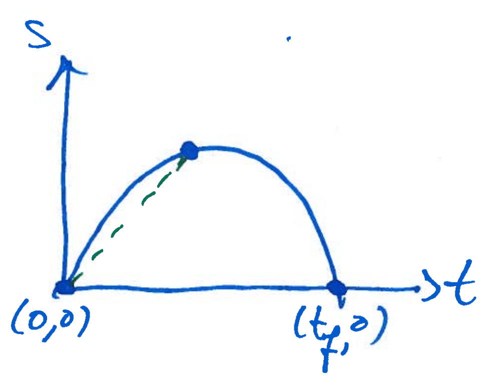
Eg: If an object leaves your hand, thrown straight up, with initial velocity, $v_0 \geq 0$, and no forces other than gravity, acceleration $-9.8\,m/s^2$, act on the object, then the height (position) relative to your hand is modeled by the function

$$s(t) = -9.8t^2 + V_0 t$$

What is the speed of the object at a given time, t_1?

- The answer is: you need calculus to answer this question.

The graph of s looks like



The average rate of change represents some sort of estimate to the speed.

2.2: Line between two points.

Two points (x_0, y_0) and (x_1, y_1) determine a unique line.

Compute the slope:

$$m = \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_0 - y_1}{x_0 - x_1}$$

$$\left(\frac{y_1 - y_0 = -(y_0 - y_1)}{x_1 - x_0 = -(x_0 - x_1)} \right)$$

Eg: $(1, 5)$, $(7, 3)$

$$\frac{3-5}{7-1} = \frac{-2}{6} = -\frac{1}{3}.$$

Compute the y-intercept, b: Solve one of the two equations

$$y_0 = mx_0 + b \quad \text{or} \quad y_1 = mx_1 + b$$

for b.

E.g.:

$$5 = \left(-\tfrac{1}{3}\right)(1) + b \qquad \text{or} \qquad 3 = \left(-\tfrac{1}{3}\right)(7) + b$$

$$= -\tfrac{1}{3} + b \qquad\qquad = \tfrac{-7}{3} + b$$

$$\frac{15}{3} = \frac{-1}{3} + b \qquad\qquad \frac{9}{3} = \frac{-7}{3} + b$$

$$\frac{15}{3} + \frac{1}{3} = \frac{16}{3} = b \qquad\qquad \frac{9+7}{3} = \frac{16}{3} = b.$$

The form $f(x) = mx + b$ is usually called Slope-Intercept Form of a line.

The point-Slope form of a line

Have: (x_0, y_0) and m

$$y - y_0 = m(x - x_0)$$

Get back to slope-intercept form by rearranging the equation

$$y - y_0 = mx - mx_0$$

$$\Rightarrow y = mx - mx_0 + y_0$$

$$= mx + \underbrace{(y_0 - mx_0)}_{b}$$

E.g.: $m = -\frac{1}{3}$, $(1,5)$

$$y - 5 = -\frac{1}{3}(x-1) = -\frac{1}{3}x + \frac{1}{3}$$

$$\Rightarrow y = -\frac{1}{3}x + \frac{1}{3} + 5$$

$$= -\frac{1}{3}x + \frac{1}{3} + \frac{15}{3}$$

$$= -\frac{1}{3}x + \frac{16}{3}.$$

Classification of Lines: $y = mx + b$

$m = 0$ $\quad y = b$, horizontal lines

$m > 0$ $\quad$ with varying degrees of steepness
the larger the m, the steeper the line

$m < 0$ $\quad$ same
the larger the m in absolute value, the steeper the line