Rational Expressions

9/7/16 ①.

A rational expression is just a ratio of two polynomials.

E.g.:

$$\frac{x^2+1}{x^2+x+1}$$

Simplify Rational Expressions

As an ~~ala~~ analogy, for ~~two~~ a rational number

$$\frac{a\cdot b}{a\cdot c} = \frac{b}{c}$$

E.g.: $\frac{4}{6} = \frac{\cancel{2}\cdot 2}{\cancel{2}\cdot 3} = \frac{2}{3}$.

To simplify a rational expression of the form

$$\frac{\cancel{A}\cdot B}{\cancel{A}\cdot C} = \frac{B}{C}.$$

E.g.: $\frac{x^2-1}{x^2+x-2} = \frac{\cancel{(x-1)}(x+1)}{(x+2)\cancel{(x-1)}} = \frac{x+1}{x+2}$.

Multiply/Divide Rational Expressions

$$\frac{A}{B}\cdot\frac{C}{D} = \frac{AC}{BD}$$

$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B}\cdot\frac{\cancel{B}D}{C} = \frac{AD}{BC}.$$

Eg: $$\frac{x^2+2x-3}{x^2+8x+16}\cdot\frac{3x+12}{x-1}=\frac{(x+3)(x-1)(3)(x+4)}{(x+4)^2(x-1)}$$

$$=\frac{3(x+3)}{x+4}$$

Eg.: $$\frac{x-4}{x^2-4}\div\frac{x^2-3x-4}{x^2+5x+6}=\frac{x-4}{x^2-4}\left(\frac{x^2+5x+6}{x^2-3x-4}\right)$$

$$=\frac{\cancel{(x-4)}\cancel{(x+2)}(x+3)}{\cancel{(x+2)}(x-2)\cancel{(x-4)}(x+1)}$$

$$=\frac{x+3}{(x-2)(x+1)}.$$

Adding and Subtracting Rational Expressions

Analogue: If we have two rational numbers, a/b, c/d, then

$$\frac{a}{b}\pm\frac{c}{d}=\frac{ad}{bd}\pm\frac{bc}{bd}=\frac{ad\pm bc}{bd}.$$

Eg: $$\frac{2}{3}+\frac{5}{7}=\frac{14}{21}+\frac{15}{21}=\frac{29}{21}.$$

For two rational expressions, A/B, C/D, then

$$\frac{A}{B}\pm\frac{C}{D}=\frac{AD\pm BC}{BD}.$$

Eg: $$\frac{3}{x-1}+\frac{x}{x+2}=\left(\frac{x+2}{x+2}\right)\left(\frac{3}{x-1}\right)+\frac{(x-1)x}{(x-1)(x+2)}$$

$$=\frac{3x+6+x^2-x}{(x-1)(x+2)}$$

$$=\frac{x^2+2x+6}{(x-1)(x+2)}$$ ← this is irreducible because the discriminant is $2^2-4(1)(6)=4-24=-20<0$.

Eg:
$$\frac{1}{x^2-1} - \frac{2}{(x+1)^2} = \frac{1}{(x+1)(x-1)} - \frac{2}{(x+1)(x+1)}$$

$$= \left(\frac{x+1}{x+1}\right)\frac{1}{(x+1)(x-1)} - \frac{(x-1)}{(x-1)}\frac{2}{(x+1)(x+1)}$$

$$= \frac{x+1-2(x-1)}{(x-1)(x+1)^2}$$

$$= \frac{x+1-2x+2}{(x-1)(x+1)^2} = \frac{-x+3}{(x-1)(x+1)^2}$$

Brute force (is bad)

$$\frac{1}{x^2-1} - \frac{2}{(x+1)^2} = \frac{(x+1)^2}{(x+1)^2}\frac{1}{x^2-1} - \frac{2}{(x+1)^2}\frac{(x^2-1)}{(x^2-1)}$$

$$= \frac{(x+1)^2 - 2(x^2-1)}{(x+1)^2(x^2-1)} = \frac{x^2+2x+1-2x^2+2}{(x+1)^2(x^2-1)}$$

$$\cancel{= \frac{x^2+2x+1-2x^2+2}{(x+1)^2(x^2-1)}} \qquad = \frac{-x^2+2x+3}{(x+1)^2(x^2-1)}$$

$$\cancel{= \frac{x^2+3}{(x+1)^2(x^2-1)}} \qquad = \frac{-(x^2-2x-3)}{(x+1)^2(x^2-1)}$$

$$= -\frac{(x-3)\cancel{(x+1)}}{(x+1)^2\cancel{(x+1)}(x-1)}$$

$$= \frac{-x+3}{(x+1)^2(x-1)}.$$

Rationalizing a Denominator

E.g.: Rationalize $\frac{1}{1+\sqrt{2}}$ by multiplying by $\frac{1-\sqrt{2}}{1-\sqrt{2}}$; one calls $1-\sqrt{2}$ the conjugate of $1+\sqrt{2}$.

$$\left(\frac{1}{1+\sqrt{2}}\right)\left(\frac{1-\sqrt{2}}{1-\sqrt{2}}\right) = \frac{1-\sqrt{2}}{1-\sqrt{2}+\sqrt{2}-2} = \frac{1-\sqrt{2}}{1-2} = \frac{1-\sqrt{2}}{-1} = \sqrt{2}-1$$

$$\frac{1-\sqrt{2}}{-1} = \left(\frac{-1}{-1}\right)\frac{1-\sqrt{2}}{-1} = \frac{-1+\sqrt{2}}{1} = \sqrt{2}-1$$

E.g.: $\frac{2}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$.

One could also multiply by the conjugate of $0+\sqrt{3}$, which is $0-\sqrt{3}$ on the top and bottom

$$\frac{2}{\sqrt{3}}\left(\frac{0-\sqrt{3}}{0-\sqrt{3}}\right) = \frac{-2\sqrt{3}}{-3} = \frac{2\sqrt{3}}{3}.$$

Long Division

Say we want to divide 23 by 4

$$\begin{array}{r} 5 \\ 4\,\overline{)\,23} \\ \underline{-20} \\ 3 \end{array}$$

This tells us that

$$23 = 4(5)+3. \quad \left(\text{equivalently } \frac{23}{4} = 5+\frac{3}{4}\right)$$

Eg.:

$$\begin{array}{r|l} & 6x - 2 \\ \hline x-4 & 6x^2 - 26x + 12 \\ & -6x^2 + 24x \\ \hline & 0 \quad -2x + 12 \\ & \quad +2x - 8 \\ \hline & \quad 0 + 4 \end{array}$$

$(x-4)6x = 6x^2 - 24x$

$(x-4)(-2) = -2x + 8$

$$6x^2 - 26x + 12 = (x-4)(6x-2) + 4$$

Check: $6x^2 - 2x - 24x + 8 + 4 = 6x^2 - 26x + 12$ ✓

So dividing both sides by $x-4$ gives

$$\frac{6x^2 - 26x + 12}{x-4} = 6x - 2 + \frac{4}{x-4}$$

Eg.: $x^3 - 1$ - want to factor, equiv. to finding solutions to $x^3 - 1 = 0$.

$1^3 - 1 = 0$, & 1 is a solution, so $x-1$ must divide x^3-1.

$$\frac{x^3-1}{x-1} = ?$$

$$\begin{array}{r|l} & x^2 + x + 1 \\ \hline x-1 & x^3 - 1 \\ & -x^3 + x^2 \\ \hline & x^2 - 1 \\ & -x^2 + x \\ \hline & x - 1 \\ & -x + 1 \\ \hline & 0 \end{array}$$

$x^2(x-1) = x^3 - x^2$

$x(x-1) = x^2 - x$

$1(x-1) = x - 1$

$$x^3 - 1 = (x-1)(x^2+x+1).$$

irreducible

$x^3 - a^3 = (x-a)(x^2+ax+a^2)$ (Difference of Two Cubes)

Check: $x^3 + \cancel{ax^2} - \cancel{ax^2} - \cancel{a^2x} + \cancel{a^2x} - a^3$

Solving equations w/ rational expressions

E.g.: $\dfrac{x}{x^2-1} = 7$

Clear fractions:

$$\left(\frac{x^2-1}{\cancel{x^2-1}}\right)\frac{x}{x^2-1} = 7(x^2-1)$$

$\Rightarrow \quad x = 7x^2 - 7$

Subtract x from both sides

$$7x^2 - x - 7 = 0$$

Disc: $(-1)^2 - 4(7)(-7) = 1 + 4(49)$
$= 1 + 196$
$= 197$

$$x = \frac{-(-1) \pm \sqrt{197}}{2(7)} = \frac{1 \pm \sqrt{197}}{14}$$

Check that your solution is not a zero of the denominator.