C1 Solving Linear Equations

For a polynomial in the variable x

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

$a_1, a_2, \ldots, a_n$ - coefficients, they're just numbers
the number n is an integer and is called the degree.

A linear polynomial/function is just a polynomial of degree one:

$$ax + b$$

Eg: $3x+2$, $7x+\pi$, $ex+2$

A linear equation has the form

degree one polynomial = number

Formally

$$ax + b = c$$

Usually one just writes

$$ax + b = 0$$

Eg: $3x+2 = 7$, could also rewrite this as

$$3x + 2 - 7 = 7 - 7$$

$$\Rightarrow 3x - 5 = 0$$

To solve an equation of the form

$$ax + b = 0$$

asks the question

"What values of x make the expression

$$ax+b=0$$

true?"

For a linear equation, there is exactly one value of x making this statement true:

when $a \neq 0$, subtract b from both sides

$$ax = -b$$

and divide both sides by a to get

$$x = -\frac{b}{a}.$$

Eg: $3x+2=0$
$\Rightarrow 3x = -2$
$\Rightarrow x = -2/3$

$7x + \pi = 0$
$\Rightarrow 7x = -\pi$
$\Rightarrow x = -\pi/7$

Eg: $\frac{x}{6} + \frac{2}{3} = \frac{3}{4}x$

Put the x's on one side : subtract $\frac{x}{6}$ from both sides

$$\frac{2}{3} = \frac{3}{4}x - \frac{x}{6} = \left(\frac{3}{4} - \frac{1}{6}\right)x = \left(\frac{9}{12} - \frac{2}{12}\right)x = \frac{7}{12}x$$

Multiply both sides by $12/7$ (equiv. divide both sides by $7/12$)

$$x = \frac{2}{3}\left(\frac{12}{7}\right) = \frac{8}{7}.$$

What if we want to let the degree of the polynomial be 2? The book refers to an equation of the form (3)

$$ax^2+bx+c=0$$

as a quadratic equation.

Easy (ish) Cases

When $b=0=c$, $ax^2=0$. Only solution is $x=0$.

When $b=0$, $ax^2+c=0$. Subtract c from both sides,

$$ax^2=-c$$

Divide both sides by a

$$x^2=\frac{-c}{a}$$

If $-c/a<0$, no real solutions. Otherwise there are 2.

Eg: $3x^2+5=0$: subtract 5 from both sides

$$3x^2=-5$$

divide both sides by 3

$$x^2=-5/3$$

No real solutions.

Eg: $x^2-1=0$

Add 1 to both sides

$$x^2=1$$

Take the two roots

$$x=1,\ x=-1.$$

Factor $x^2-1^2=(x-1)(x+1)=0$

ZFP says either $x-1=0$ or $x+1=0$

So either $x=1$ or $x=-1$.

$ax^2+bx+c=0$

Completing the Square

Recall: $(x+a)^2 = x^2+2ax+a^2$

$(x-a)^2 = x^2-2ax+a^2$

$x^2+5x+6=0$

$x^2+5x+6 = x^2+2(\frac{5}{2})x+6=0$

Add $(5/2)^2$ to both sides

$$x^2+5x+6+(5/2)^2 = \underbrace{x^2+2(5/2)x+(5/2)^2}+6 = (5/2)^2$$

$$=(x+5/2)^2+6=(5/2)^2$$

Solving $x^2+5x+6=0$ is equivalent to solving

$$(x+5/2)^2+6=(5/2)^2$$

$$\Rightarrow (x+5/2)^2 = \frac{25}{4}-6 = 6+\frac{1}{4}-6=\frac{1}{4}$$

$$\Rightarrow (x+5/2) = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$$

$$\Rightarrow x = -5/2 \pm 1/2$$

$x = -5/2+1/2$ and $x=-5/2-1/2$

$= -4/2$ $\quad\quad$ $= -6/2$

$= -2$ $\quad\quad$ $= -3$

Quadratic Formula

The solutions to

$$ax^2+bx+c=0$$

are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Eg: $1 \cdot x^2 + 5x + 6 = 0$

$$x = \frac{-5 \pm \sqrt{\cancel{2}(5)^2 - 4(1)(6)}}{2(1)}$$

$$= \frac{-5 \pm \sqrt{25 - 24}}{2}$$

$$= \frac{-5 \pm \sqrt{1}}{2} = \frac{-5 \pm 1}{2}$$

$$x = \frac{-5 - 1}{2} = \frac{-6}{2} = -3 \quad \text{or} \quad x = \frac{-5+1}{2} = \frac{-4}{2} = -2.$$

Eg: Factor $x^2 + 7x + 12$

Solve $x^2 + 7x + 12 = 0$

$$\frac{-7 \pm \sqrt{7^2 - 4(1)(12)}}{2(1)} = \frac{-7 \pm \sqrt{49 - 48}}{2} = \frac{-7 \pm 1}{2}$$

$$x = \frac{-7+1}{2} = \frac{-6}{2} = \underline{-3} \quad \text{or} \quad x = \frac{-7-1}{2} = \frac{-8}{2} = \underline{-4}.$$

$$(x - (-3))(x - (-4)) = (x+3)(x+4) = x^2 + 4x + 3x + 12$$

$$= x^2 + 7x + 12.$$

Factor $6x^2-7x-5$

Solve $6x^2-7x-5=0$

$$x = \frac{-(-7)\pm\sqrt{(-7)^2-4(6)(-5)}}{2(6)} = \frac{7\pm\sqrt{49+120}}{12}$$

$$\begin{array}{r} 13 \\ 13 \\ \hline 39 \\ 130 \\ \hline 169 \end{array}$$

$$= \frac{7\pm\sqrt{169}}{12}$$

$$= \frac{7\pm 13}{12}$$

$$x = \frac{7+13}{12} = \frac{20}{12} = \frac{5\cdot\cancel{4}}{\cancel{4}\cdot 3} = \frac{5}{3} \qquad x = \frac{7-13}{12} = \frac{-6}{12} = -\frac{1}{2}$$

$$(x-5/3)(x-(\tfrac{-1}{2})) = (x-5/3)(x+1/2)$$

$$= x^2+\tfrac{1}{2}x - 5/3x - 5/6$$

$$= x^2+\tfrac{3}{6}x-\tfrac{10}{6}x-5/6$$

$$= x^2 - 7/6x - 5/6$$

Multiply both sides by 6:

$$6x^2-7x-5 = 6(x-5/3)(x+1/2) = 2\cdot 3(x-5/3)(x+1/2)$$

$$= 3(x-5/3)(2)(x+1/2)$$

$$= (3x-5)(2x+1).$$

In the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

(which gives solutions to $ax^2+bx+c=0$)

b^2-4ac is called the discriminant. There are three interesting values:

$b^2-4ac=0$:

$$\Rightarrow x = \frac{-b \pm \sqrt{0}}{2(a)} = \frac{-b}{2a}$$

This says ax^2+bx+c is a perfect square!

$b^2-4ac>0$: this says there are 2 real solutions to

$$ax^2+bx+c=0$$

$b^2-4ac<0$: There are no real solutions to

$$ax^2+bx+c=0.$$

Eg: $x^2+1=0$

Disc: $0^2-4(1)(1)=-4$

Eg: $x^2+4x+4=0$

Disc: $(4)^2-4(1)(4)=16-16=0.$

Solution: $x=\frac{-4}{2(1)}=-2$

$$(x-(-2))(x-(-2))=(x+2)(x+2)=(x+2)^2=x^2+4x+4.$$

Eg: $x^2+5x+6=(x+2)(x+3)$

Disc: $5^2-4(1)(6)=25-24=1$

Already know solutions are $x=-3$, $x=-2$.