B.2 Factoring Algebraic Expressions

8/31/17

E.g: Factoring out common factors

a) $3x^2 - 6x = 3x(x-2)$

Check: $(3x)(x) - (3x)(2) = 3x^2 - 6x.$

b) $8x^4y^2 + 6x^3y^3 - 2xy^4 = 2xy^2(4x^3 + 3x^2y - y^2)$

E.g: $(2x+4)(x-3) - 5(x-3) = (x-3)(2x+4-5)$
$= (x-3)(2x-1)$

Factoring Trinomials

A trinomial is a polynomial with three terms

$$ax^2 + b\overset{x}{\cancel{y}} + c$$

Effectively, we want to write such an ~~exp~~ expression as $x^2 + bx + c$ into something of the form

$$(x-r)(x-s)$$

Expanding this we see that

$$(x-r)(x-s) = x^2 - sx - rx + rs$$
$$= x^2 - (s+r)x + rs$$

If $(x-r)(x-s) = x^2 + bx + c$, then $-b = s+r$, $c = rs$.

Eg: $x^2 + 7x + 12 = (x+3)(x+\overset{4}{\cancel{7}}) = x^2 + 4x + 3x + 12 = x^2 + 7x + 12.$

E.g: $6x^2+7x-5$

$$(2x-1)(3x+5) = 6x^2+10x-3x-5$$
$$= 6x^2+7x-5$$

E.g: $x^2-2x-3 = (x-3)(x+1)$

$$(x-3)(x+1) = x^2+x-3x-3$$
$$= x^2-2x-3 \checkmark$$

E.g.: $(5a+1)^2-2(5a+1)-3$

Change variables to make the expression more familiar!

Let $x = 5a+1$

Rewrite:

$$(5a+1)^2-2(5a+1)-3 = x^2-2x-3$$
$$= (x-3)(x+1)$$

Substitute the $5a+1$ back into the factored expression

$$(5a+1)^2-2(5a+1)-3 = (x-3)(x+1)$$
$$= ((5a+1)-3)((5a+1)+1)$$
$$= (5a-2)(5a+2)$$

E.g: Factor

$$x+5\sqrt{x}+6$$

Let $y=\sqrt{x}$, so $y^2=(\sqrt{x})^2 = x$

$$x+5\sqrt{x}+6 = y^2+5y+6 = (y+2)(y+3) = (\sqrt{x}+2)(\sqrt{x}+3)$$

$x^4 + 7x^2 + 12$

Let $y = x^2$, $y^2 = (x^2)^2 = x^{2 \cdot 2} = x^4$

$$x^4 + 7x^2 + 12 = y^2 + 7y + 12$$
$$= (y+3)(y+4)$$
$$= (x^2+3)(x^2+4)$$

both of these cannot be reduced any further.
They are "irreducible".

E.g: $x^2 - 7x^2 + 12$

$$x^2 - 7x^2 + 12 = (x^2-3)(x^2-4)$$
$$= (x+\sqrt{3})(x-\sqrt{3})(x+2)(x-2)$$

Check: $(x+\sqrt{3})(x-\sqrt{3}) = x^2 - \sqrt{3}x + \sqrt{3}x \; \sqrt{3}(-\sqrt{3})$
$$= x^2 - 3$$
$$(x+2)(x-2) = x^2 - 2x + 2x - 4 = x^2 - 4.$$

In general,

Difference of two squares

$$x^2 - a^2 = (x-a)(x+a)$$

Perfect Squares: $(x+a)^2 = x^2 + 2ax + a^2$
$$(x-a)^2 = x^2 - 2ax + a^2$$

E.g.: Factor

a) $4x^2-25$

b) $(x+y)^2-z^2=((x+y)+z)((x+y)-z)$

$$4x^2-25=2^2x^2-5^2=(2x)^2-5^2$$
$$=(2x+5)(2x-5)$$

E.g: $x^6-9=x^{3\cdot 2}-9=(x^3)^2-3^2=(x^3+3)(x^3-3)$

E.g: $x^2+6x+9=x^2+2(3)x+(3)^2$
$$=(x+3)^2$$

E.g: $4x^2-4xy+y^2=2^2x^2-2(2x)y+y^2$
$$=(2x)^2-2(y)(2x)+(y)^2$$
$$=(2x-y)^2$$

Alternatively: $y^2-2(2x)y+(2x)^2=(y-2x)^2$

These are the same:

$$(y-2x)^2=(-1(2x-y))^2=(-1)^2(2x-y)^2=(2x-y)^2$$

Factoring by Grouping

E.g: $x^3+x^2+4x+4=x^2(x+1)+4(x+1)$
$$=(x+1)(x^2+4)$$

E.g: $x^3-2x^2-3x+6=x^2(x-2)-3(x-2)$
$$=(x-2)(x^2-3)=(x-2)(x-\sqrt{3})(x+\sqrt{3}).$$

Fundamental Theorem of Algebra

If you ~~to~~ allow complex numbers $(a+\underbrace{(\sqrt{-1})}_{i}b)$, then every polynomial

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

factors as

$$(x-r_1)(x-r_2)\cdots(x-r_n)$$

where the r_i's may not be distinct.

Eg: $x^2-1=(x+1)(x-1) \Leftrightarrow$ the two solutions to x^2-1 are $1,-1$

$$x^2+1=(x+\sqrt{-1})(x-\sqrt{-1}) = x^2 - \cancel{\sqrt{-1}x} \cancel{+\sqrt{-1}x} - (\sqrt{-1})^2$$

$$= x^2 - (-1) = x^2+1.$$

$$x^2+2ax+a^2 = (x+a)^2 = (x+a)(x+a)$$

$$x^2-2ax+a^2 = (x-a)^2 = (x-a)(x-a)$$

C.Z

Zero-Product Property

For any two real numbers,

$$AB = 0 \underset{\text{"if and only if"}}{\Longleftrightarrow} A=0 \text{ or } B=0 \text{ or both.}$$

This can be translated to saying solving an equation of the form

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

is equivalent to factoring this polynomial.