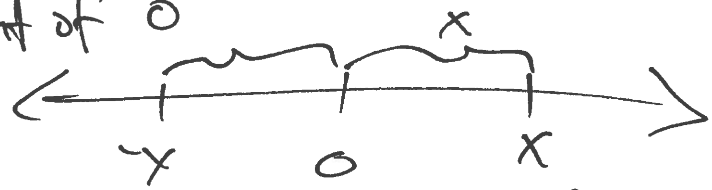
Absolute Value & Distance

8/29/17 (1)

Def$^{\underline{n}}$: The absolute value of a number x is

$$|x| = \begin{cases} x & \text{if } 0 \le x, \\ -x & \text{if } x < 0. \end{cases}$$

Reason: If $x > 0$, by definition, x lies x units to the right of 0



$-x$ lies to the left of 0 by x units.

$|x|$ just asks the question

"How far away from the origin is the number x?"

The distance function is defined to be

$$d(a,b) = |b-a|$$

It asks the question

"How far away from b is a?"

or, symmetrically,

"How far away from a is b?"

Algebraically

$$d(a,b) = |b-a| = |-(a-b)| = |-1|\,|a-b| = |a-b| = d(b,a).$$

Examples

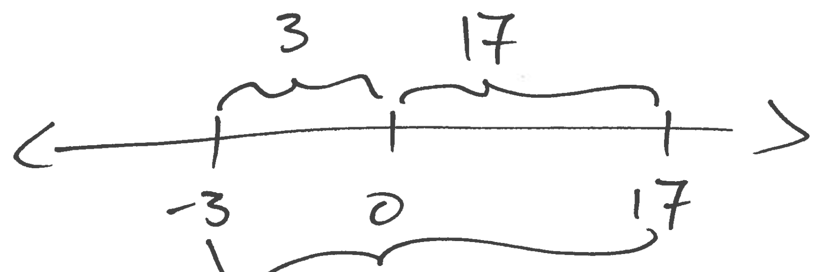
$|3| = 3, \quad |-3| = -(-3) = 3.$ etc.

Rmk: $|3| = d(0,3) = d(3,0) = |3-0| = |3| = 3.$

$|-3| = d(0,-3) = d(-3,0) = |-3-0| = |-3| = 3.$

Alg $d(-3,17) = |17-(-3)| = |17+3| = |20| = 20.$

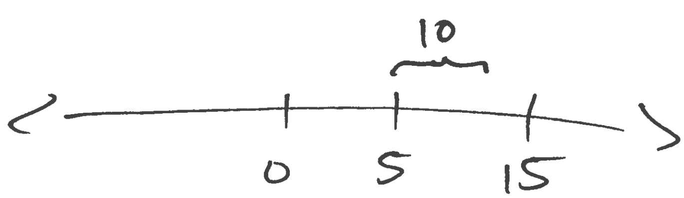
Geo



$d(-3,17) = 3+17 = 20.$

$d(5,15)$

Geo



Alg $d(15,5) = d(5,15) = |15-5| = |10| = 10.$

A3 Integer Exponents

Exponential Notation

If a is a real number, $(a \in \mathbb{R})$, n is a positive integer ~~$(n\in\mathbb{Z})$~~ $(n \in \mathbb{N})$

$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ times.}}$$

E.g: $2^3 = 2 \cdot 2 \cdot 2$

$$\pi^7 = \pi \cdot \pi \cdot \pi \cdot \pi \cdot \pi \cdot \pi \cdot \pi$$

Rules for Exponents

$a^0 = 1, \ a \neq 0$

$a^{-1} = \frac{1}{a}, \quad a^{-n} = (a^{-1})^n = \left(\frac{1}{a}\right)^n = \frac{1}{a^n}$

$a^m a^n = a^{m+n}$

$$\underbrace{(aaa \cdots a)}_{m}\underbrace{(aaa \cdots a)}_{n} = \underbrace{a \cdot a \cdot a \cdots a}_{m+n}$$

E.g: $3^2 3^4 = 3^6$

$$\underbrace{(3 \cdot 3)}_{2}\underbrace{(3 \cdot 3 \cdot 3 \cdot 3)}_{4} = \underbrace{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}_{6}$$

$\frac{a^m}{a^n} = a^{m-n}$

$$\frac{\overbrace{a \cdot a \cdots a}^{m}}{\underbrace{a \cdot a \cdots a}_{n}} = a^{m-n}$$

E.g: $4^5/4^6 = \frac{\cancel{4}\cdot\cancel{4}\cdot\cancel{4}\cdot\cancel{4}\cdot\cancel{4}}{\cancel{4}\cdot\cancel{4}\cdot\cancel{4}\cdot\cancel{4}\cdot\cancel{4}\cdot 4} = \frac{1}{4} = 4^{-1} = 4^{5-6}$

$$\frac{4^6}{4^5} = \frac{\cancel{4}\cdot\cancel{4}\cdot\cancel{4}\cdot\cancel{4}\cdot\cancel{4}\cdot 4}{\cancel{4}\cdot\cancel{4}\cdot\cancel{4}\cdot\cancel{4}\cdot\cancel{4}} = 4^1 = 4^{6-5}$$

$$(a^m)^n = a^{mn}$$

$$\overbrace{(\underbrace{aaa\cdots a}_{m})(\underbrace{a\cdot a\cdot a\cdots a}_{m})\cdots(\underbrace{a\cdot a\cdot a\cdots a}_{m})}^{n}$$

E.g: $(3^2)^3 = 3^6$

$$\overbrace{(\underbrace{3\cdot 3}_{2})(\underbrace{3\cdot 3}_{2})(\underbrace{3\cdot 3}_{2})}^{3} = \underbrace{3\cdot 3\cdot 3\cdot 3\cdot 3\cdot 3}_{6}$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$\left(\frac{a}{b}\right)^n = \underbrace{\frac{a}{b}\cdot\frac{a}{b}\cdot\frac{a}{b}\cdots\frac{a}{b}}_{n} = \frac{a^n}{b^n}$$

E.g: $\left(\frac{2}{3}\right)^3 = \frac{2^3}{3^3} = \frac{8}{27}$

$$\left(\frac{2}{3}\right)^3 = \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right) = \frac{2\cdot 2\cdot 2}{3\cdot 3\cdot 3} = \frac{2^3}{3^3}$$

A.4 Radicals and Rational Exponents

We want to extend exponents to the rational numbers

$$\mathbb{Q} = \{\tfrac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{Z}\} \quad (\text{E.g: } 2/3, 5/2, 7, 21/2)$$

We start with $\frac{1}{n}$ for n some positive integer, we define

$a^{1/n}$ to be a real number such that

$$(a^{1/n})^n = a.$$

We give this a special symbol:

$$a^{1/n} = \sqrt[n]{a}$$

When ~~a~~ n is even, we require that $a \geq 0$.

Heuristic reason: suppose $a = -2$. The number $\sqrt{-2} = (-2)^{1/2}$ is a number such that

$$\sqrt{-2}\sqrt{-2} = (-2)^{1/2}(-2)^{1/2} = -2$$

But $\sqrt{-2}$ can't be negative because

$$(\sqrt{-2})(\sqrt{-2}) \neq -2$$

would be positive and can't be positive, because then

$$(\sqrt{-2})(\sqrt{-2}) \neq -2$$

positive

When n is odd, a can be negative.

Eg:

$$\sqrt[3]{-8} = (-8)^{1/3}$$

This asks

"What is a number that when multiplied by itself three times is -8?"

The answer is -2:

$$(-2)(-2)(-2) = (-1)^3 2^3 = -8$$

Even worse for even roots: there are two answers.

Eg What is $\sqrt[4]{16}$? This should be a number such that

$$(\sqrt[4]{16})^4 = (16^{1/4})^4 = 16.$$

There's an obvious answer: 2

$$2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$$

and a less obvious answer: -2

$$(-2)^4 = (-2)(-2)(-2)(-2) = 2^4 = 16$$

To settle this ambiguity, we define $\sqrt[n]{a}$ to be the <u>positive</u> number such that

$$(\sqrt[n]{a})^n = a$$

when n is <u>even</u>.

For $m, n \in \mathbb{Z}$, $m/n \in \mathbb{Q}$, define

$$a^{m/n} = (a^{1/n})^m = (a^m)^{1/n}$$

Radicals:

$$\left(\sqrt[n]{a}\right)^m = \sqrt[n]{(a^m)}$$

E.g.: ① $16^{3/4} = (16^{1/4})^3 = 2^3 = 8$

② $(-8)^{2/3} = ((-8)^{1/3})^2 = (-2)^2 = 4$

Alternatively: $((-8)^2)^{1/3} = 64^{1/3} = (2^6)^{1/3} = 2^{6/3} = 2^2 = 4.$

③ $\left(\frac{1}{32}\right)^{-3/5} = \left(\left(\frac{1}{32}\right)^{-1}\right)^{3/5}$

$= (32)^{3/5} = ((2^5)^{1/5})^3 = (2^1)^3 = 8.$

④ $\left(\frac{8}{27}\right)^{4/3} = \left(\frac{2^3}{3^3}\right)^{4/3} = \left(\frac{2^{3/3}}{3^{3/3}}\right)^4 = \left(\frac{2}{3}\right)^4 = \frac{16}{81}$

$$\left(\frac{2^3}{3^3}\right)^{4/3} = \left(\frac{2^3}{3^3}\right)^{(1/3)\cdot 4} = \left(\left(\frac{2^3}{3^3}\right)^{1/3}\right)^4$$

$$= \left(\frac{2^{3\cdot 1/3}}{3^{3\cdot 1/3}}\right)^4 = \left(\frac{2^1}{3^1}\right)^4$$

$$= \left(\frac{2}{3}\right)^4$$